TUTORIAL 9 SOLUTIONS (NO DATE) – VERSION 1

Problem 3, Section 16.4 (Adams)

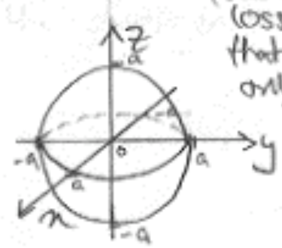
Question: Find the outward flux of $\vec{F} = (x^2+y^2)\hat{\imath} + (y^2-z^2)\hat{\jmath} + z\hat{k}$ through sphere $S: x^2+y^2+z^2=a^2$

Solution:

Let V be the region bounded by S; then by the divergence theorem:

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_V (\text{div}\,\vec{F})\,dV$$

where $d\vec{S}$ is taken out of S



(assume without loss of generality that $a>0$, since only a^2 appears in the problem)

Evaluate the volume integral using the method from Tutorial #2:

1) Sketch the volume V: see above
2) Pick a coordinate system: spherical
3) Parametrize the volume V: $0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$
4) Integrand:

$$\text{div}\,\vec{F} = \partial F_1/\partial x + \partial F_2/\partial y + \partial F_3/\partial z = 2x + 2y + 1 = 2\rho\sin\varphi\cos\theta + 2\rho\sin\varphi\sin\theta + 1$$

5) $dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$
6) Evaluate the integral:

$$\iiint_V (\text{div}\,\vec{F})\,dV = \int_0^a d\rho \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi\,[2\rho\sin\varphi\cos\theta + 2\rho\sin\varphi\sin\theta + 1](\rho^2\sin\varphi)$$

$$= \int_0^a \rho^2\,d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,d\varphi$$

$$= [\rho^3/3]_0^a \cdot 2\pi \cdot [-\cos\varphi]_0^{\pi}$$

$$= (a^3/3)\cdot 2\pi \cdot [-(-1)+1] = 4\pi a^3/3$$

Note: the terms with $\cos\theta$ and $\sin\theta$ integrate out to zero: $\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0$

$\therefore$ the outward flux is $\boxed{4\pi a^3/3}$

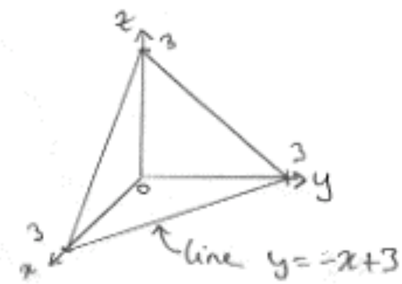
TUTORIAL 9 SOLUTIONS (NO DATE) – VERSION 1

Problem 7, Section 16.4 (Adams)

Question: Same as Problem 3 with $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ and S being the boundary of the tetrahedron $(x+y+z \leq 3) \wedge (x,y,z \geq 0)$

Solution:

Again let V be the region bounded by S, apply the divergence theorem and compute $\iiint_V (\operatorname{div} \vec{F})\, dV$ (instead of four flux integrals...)



1) Sketch V: see the figure

2) Use Cartesian (3D)

3) Parametrize V: by inspecting the figure, $0 \leq x \leq 3$, $0 \leq y \leq -x+3$, $0 \leq z \leq 3-x-y$

4) Integrand: $\operatorname{div}\vec{F} = 2x+2y+2z$

5) $dV = dx\, dy\, dz$

6) $\iiint_V (\operatorname{div}\vec{F})\, dV = \int_0^3 dx \int_0^{3-x} dy \int_0^{3-x-y} dz\, [2x+2y+2z]$

$= 2\cdot\int_0^3 dx \int_0^{3-x} \left[xz + yz + z^2/2\right]_{z=0}^{3-x-y} dy$

$= 2\cdot\int_0^3 dx \int_0^{3-x} \left[x(3-x-y) + y(3-x-y) + (3-x-y)^2/2\right] dy$

$= 2\cdot\int_0^3 dx \int_0^{3-x} \left[3x - x^2 - xy + 3y - xy - y^2 + (9 - 2(3)(x+y) + (x+y)^2)/2\right] dy$

$= \int_0^3 dx \int_0^{3-x} \left[6x + 6y - 4xy - 2x^2 - 2y^2 + (9 - 6x - 6y + x^2 + 2xy + y^2)\right] dy$

$= \int_0^3 dx \int_0^{3-x} \left[9 - x^2 - 2xy - y^2\right] dy$

$= \int_0^3 \left[9y - x^2y - xy^2 - y^3/3\right]_{y=0}^{3-x} dx$

$= \int_0^3 \left[9(3-x) - x^2(3-x) - x(3-x)^2 - (1/3)(3-x)^3\right] dx$

$= \int_0^3 \left[27 - 9x - 3x^2 + x^3 - x(9 - 6x + x^2) - (1/3)(3^3 - 3\cdot 3^2 \cdot x + 3\cdot 3\cdot x^2 - x^3)\right] dx$

$= \int_0^3 \left[27 - 9x - 3x^2 + x^3 - 9x + 6x^2 - x^3 - 9 + 9x - 3x^2 + x^3/3\right] dx$

$= \int_0^3 \left[18 - 9x + x^3/3\right] dx$

$= \left[18x - 9x^2/2 + x^4/12\right]_0^3$

$= 18(3) - 9(3)^2/2 + 3^4/12$

$= 54 - 81/2 + 27/4$

$= 216/4 - 162/4 + 27/4$

$= 81/4$ (long integral, but easy to do...)

$\therefore$ the outward flux is $\boxed{81/4}$

TUTORIAL 9 SOLUTIONS (NO DATE) – VERSION 1

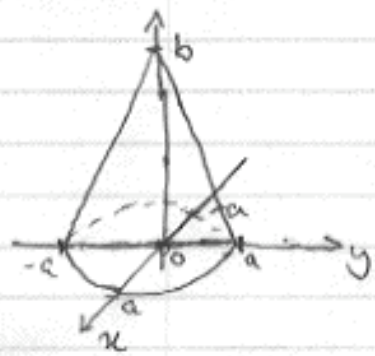
Problem 11, Section 16.4 (Adams)

A conical domain has apex $(0,0,b)$, axis along the z-axis, and a base that is a disk of radius a in the xy plane

Find the outward flux of
$\vec{F} = (x+y^2)\hat{\imath} + (3x^2y + y^3 - x^3)\hat{\jmath} + (z+1)\hat{k}$
through the conical part of the domain

Solution:

Instead of parametrizing the lateral part of the cone, find the flux out of the entire domain and subtract the flux through the base



Let V denote the domain, S the surface bounding it, S_{cone} the lateral part of the cone, and S_{base} the base (so $S = S_{base} \cup S_{cone}$)

Then
$$\iint_{S_{cone}} \vec{F}\cdot d\vec{S} = \oiint_S \vec{F}\cdot d\vec{S} - \iint_{S_{base}} \vec{F}\cdot d\vec{S}$$
$$= \iiint_V (\operatorname{div}\vec{F})\, dV - \iint_{S_{base}} \vec{F}\cdot d\vec{S}$$

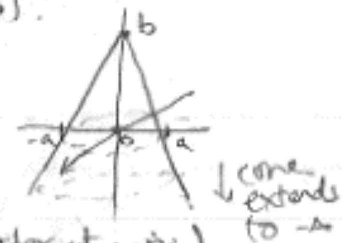
We have two integrals to evaluate, a volume and a surface one. First the volume:

1) Sketch the region V: see above
2) Coordinate system: Cylindrical (others might work too)
3) Describe (parametrize) V:

Projecting onto the xy plane shows that
$0 \le r \le a, \quad 0 \le \theta \le 2\pi$
(the projection is just the disk $x^2+y^2 \le a^2$)

What about z? Well, the lateral surface of the cone (extending all the way to $z=-\infty$). is described by the equation

$$z = b - (b/a)\sqrt{x^2+y^2} = b - br/a$$



(recall that $z=\sqrt{x^2+y^2}$ is the familiar upward-pointing cone oriented at 45° with vertex at origin)

Hence $0 \le z \le b - br/a$ and the complete parametrization for V is
$(0 \le r \le a) \wedge (0 \le \theta \le 2\pi) \wedge (0 \le z \le b - br/a)$

4) Integrand: $\operatorname{div}\vec{F} = 1 + (3x^2+3y^2) + 1 = 3r^2+2$

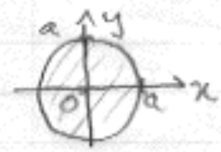
5) $dV = r\,dr\,d\theta\,dz$

6) Evaluate the integrand:

$$\begin{aligned}
\iiint_V (\operatorname{div}\vec{F})\,dV &= \int_0^a dr \int_0^{2\pi} d\theta \int_0^{b-br/a} dz\,(3r^2+2)(r) \\
&= \int_0^a (3r^3+2r)\,dr \cdot \int_0^{2\pi} d\theta \cdot \int_0^{b-br/a} dz \\
&= \int_0^a (3r^3+2r)(b-br/a)\cdot(2\pi)\,dr \\
&= 2\pi b \int_0^a (3r^3+2r-3r^4/a-2r^2/a)\,dr \\
&= 2\pi b \left[\frac{3r^4}{4} + r^2 - \frac{3r^5}{5a} - \frac{2r^3}{3a}\right]_0^a dr \\
&= 2\pi b\left[3a^4/4 + a^2 - 3a^4/5 - 2a^2/3\right] \\
&= 3\pi a^4 b/10 + 2\pi a^2 b/3
\end{aligned}$$

Now for the surface integral

1) Sketch the region S_{base}
Disk of radius a in the xy plane



2) Coordinate system: plane polar (cylindrical with $z=0$)

3) Parametrize the region:

$$\vec{r}_S(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + \underset{z=0}{0}\cdot\hat{k}$$

where $(0 \le r \le a) \wedge (0 \le \theta \le 2\pi)$

4). Integrand: $\vec{F} = (\ldots)\hat{\imath} + (\ldots)\hat{\jmath} + \hat{k}$

(note $z=0$ on S $\Rightarrow \hat{k}$ component is $z+1=1$)

we won't need these components, $d\vec{S}$ will point only in the $\hat{k}$ direction $\Rightarrow \hat{\imath}$ and $\hat{\jmath}$ parts of $\vec{F}$ die

5) $d\vec{S}$: a unit normal to the surface is $\hat{N} = \pm\hat{k}$, and also $dS = r\,dr\,d\theta\,dz$

$\Rightarrow d\vec{S} = \pm\hat{k}\, r\,dr\,d\theta\,dz$

Which direction? The divergence theorem requires outward-pointing flux, and so $d\vec{S}$ must point downward on the base.

$\therefore\ d\vec{S} = (-r\hat{k})\,dr\,d\theta$

6) $\iint_{S_{base}} \vec{F}\cdot d\vec{S} = \int_0^a dr \int_0^{2\pi} d\theta\,(-r)$

$= -(2\pi - 0)\cdot\left[r^2/2\right]_0^a$

$= -\pi a^2$

Thus $\iint_{S_{cone}} \vec{F}\cdot d\vec{S} = \oiint_S \vec{F}\cdot d\vec{S} - \iint_{S_{base}} \vec{F}\cdot d\vec{S}$

$= \dfrac{2\pi a^2 b}{3} + \dfrac{3\pi a^4 b}{10} - (-\pi a^2)$

$\therefore$ the outward flux over the conical part of the domain is $\boxed{\dfrac{2\pi a^2 b}{3} + \dfrac{3\pi a^4 b}{10} + \pi a^2 b}$

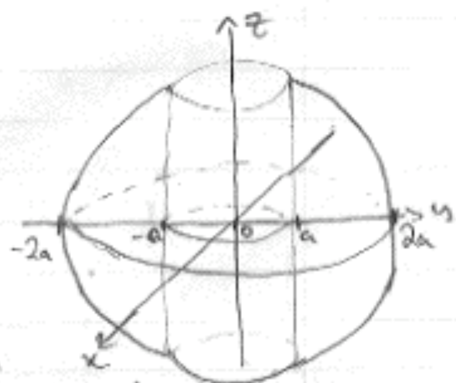
TUTORIAL 9 SOLUTIONS (NO DATE) – VERSION 1

Problem 13, Section 16.4 (Adams)

Question: Let D be the region formed by $x^2+y^2+z^2 \le 4a^2$ and $x^2+y^2 \ge a^2$. Denote the surface bounding D by S, and let S_1 denote the cylindrical part of S and S_2 the spherical part.

Find the outward flux of
$\vec{F} = (x+yz)\hat{i} + (y-xz)\hat{j} + (z-e^x \sin y)\hat{k}$
through:
a) S b) S_1 c) S_2

Part A



Use the divergence theorem:
$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_V (\text{div}\,\vec{F})\, dV$

1) Sketch the volume
2) Coordinate system: cylindrical (we have a cylinder and a sphere)
3) Parametrize V:

Sphere: $x^2+y^2+z^2 \le 4a^2 \Rightarrow r^2+z^2 \le 4a^2$
$\Rightarrow z^2 \le 4a^2 - r^2$
$\Rightarrow -\sqrt{4a^2-r^2} \le z \le \sqrt{4a^2-r^2}$

Projecting onto the xy plane shows that $a \le r \le 2a$ and $0 \le \theta \le 2\pi$

$$\therefore V: \begin{cases} a \le r \le 2a \\ 0 \le \theta \le 2\pi \\ -\sqrt{4a^2-r^2} \le z \le \sqrt{4a^2-r^2} \end{cases}$$

4) Integrand: $\text{div}\,\vec{F} = 1+1+1 = 3$
5) $dV = r\,dr\,d\theta\,dz$

6) Evaluate:

$$\iiint_V (\operatorname{div} \vec{F})\,dV = \int_a^{2a} dr \int_0^{2\pi} d\theta \int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} dz \cdot (3)(r)$$

$$= 3 \cdot \int_a^{2a} r \cdot 2\pi \cdot \left[\sqrt{4a^2-r^2} - \left(-\sqrt{4a^2-r^2}\right)\right] dr$$

$$= 12\pi \cdot \int_a^{2a} r\sqrt{4a^2-r^2}\,dr$$

$$= 12\pi \cdot \int_{3a^2}^{0} u^{1/2}\left(-du/2\right)$$

Let $u = 4a^2 - r^2$
$du = -2r\,dr$
$\Rightarrow r\,dr = -du/2$
$r = a \Rightarrow u = 3a^2$; $r = 2a \Rightarrow u = 0$

$$= 6\pi \int_0^{3a^2} u^{1/2}\,du$$

$$= 6\pi \left[\frac{u^{3/2}}{3/2}\right]_0^{3a^2}$$

$$= 4\pi \cdot (3a^2)^{3/2} = 4\pi\sqrt{27}\,a^3$$

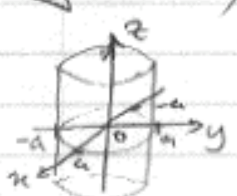
$$= 12\sqrt{3}\,\pi a^3$$

$\therefore$ outward flux of $\vec{F}$ through S is $\boxed{12\sqrt{3}\,\pi a^3}$

Part B

We want $\iint_{S_1} \vec{F} \cdot d\vec{S}$; we can't use the divergence theorem here (not a closed surface), so just do the surface integral (S_1 is just a cylinder...)

1) Sketch S_1: cylinder of radius a
2) Coordinate system: cylindrical
3) Parametrize S_1:



On the cylinder, $x^2+y^2=a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$
Also $0 \le \theta \le 2\pi$
What about z? We need to find the intersection of the cylinder with the sphere:

$$(x^2+y^2+z^2 = 4a^2) \wedge (x^2+y^2 = a^2)$$

$$\Rightarrow a^2 + z^2 = 4a^2 \Rightarrow z^2 = 3a^2$$

$$\Rightarrow z = \pm\sqrt{3}\,a$$

$$\Rightarrow -\sqrt{3}\,a \le z \le \sqrt{3}\,a$$

$\therefore \vec{r}_S(\theta, z) = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath} + z\,\hat{k}$
where $(0 \le \theta \le 2\pi) \wedge (-\sqrt{3}\,a \le z \le \sqrt{3}\,a)$

4) Integrand:

$$\vec{F} = (a\cos\theta + az\sin\theta)\hat{\imath} + (a\sin\theta - az\cos\theta\hat{\jmath}) + (\ldots)\hat{k}$$

(we won't need the $\hat{k}$ component, as $d\vec{S}$ has no $\hat{k}$ component...)

5) Find $d\vec{S}$

A unit normal to a cylinder is

$$\hat{N} = \frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}} = \frac{a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath}}{a}$$
$$= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

Also $dS = r\,d\theta\,dz = a\,d\theta\,dz$

So $d\vec{S} = \pm\hat{N}\,dS = \pm(a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath})\,d\theta\,dz$

We want $d\vec{S}$ to point out of $\underline{S}$ (not S_1!). So for, e.g., $\theta = 0$, we want a negative $\hat{\imath}$ component (for $\theta = 0$, the $\hat{\imath}$ component is $a\cos 0 = a$)

Thus we pick the "$-$" sign.

$\therefore d\vec{S} = -(a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath})\,d\theta\,dz.$

6) Evaluate the integral

$$\iint_{S_1}\vec{F}\cdot d\vec{S} = -\int_0^{2\pi} d\theta \int_{-\sqrt{3}a}^{\sqrt{3}a} dz\,(a^2\cos^2\theta + a^2z\cos\theta\sin\theta + a^2\sin^2\theta - a^2z\sin\theta\cos\theta)$$
$$= -a^2\cdot 2\pi\cdot(\sqrt{3}a - (-\sqrt{3}a))$$
$$= -4\sqrt{3}\pi a^3$$

$\therefore$ outward flux of $\vec{F}$ through S_1 is $\boxed{-4\sqrt{3}\pi a^3}$

Part C

$$\iint_{S_2}\vec{F}\cdot d\vec{S} = \oiint_S \vec{F}\cdot d\vec{S} - \iint_{S_1}\vec{F}\cdot d\vec{S} \quad (\text{since } S = S_1 \cup S_2)$$
$$= 12\sqrt{3}\pi a^3 - (-4\sqrt{3}\pi a^3)$$
$$= \boxed{16\sqrt{3}\pi a^3}$$