**TUTORIAL 7 SOLUTIONS (OCTOBER 26, 2006) – VERSION 1**

## Problem 7, Section 15.4 (Adams)

Question: Find the work done by the force field $\vec{F} = (x+y)\hat{\imath} + (x-z)\hat{\jmath} + (z-y)\hat{k}$ in moving an object from $(1,0,-1)$ to $(0,-2,3)$ along any smooth curve.

Solution:

We want $\int_C \vec{F}\cdot d\vec{r}$ where $C$ is any smooth curve from $(1,0,-1)$ to $(0,2,3)$.
The question suggests that this line integral is the same no matter the curve, which would be the case if $\vec{F}$ is conservative.
Let's try to find a potential for $\vec{F}$. Let $\vec{F} = \operatorname{grad}\varphi(x,y,z)$; then

$$\underbrace{(\partial\varphi/\partial x = x+y)}_{①} \wedge \underbrace{(\partial\varphi/\partial y = x-z)}_{②} \wedge \underbrace{(\partial\varphi/\partial z = z-y)}_{③}$$

From ①: $\varphi = \int (x+y)\,dx + k_1(y,z) = x^2/2 + xy + k_1(y,z)$ ④

From ④: $\partial\varphi/\partial y = x + \partial k_1(y,z)/\partial y$ ⑤

$\partial\varphi/\partial z = \partial k_1(y,z)/\partial z$ ⑥

Equate ② with ⑤ and ④ with ⑥:

$$\underbrace{(\partial k_1(y,z)/\partial y = -z)}_{⑦} \wedge \underbrace{(\partial k_1(y,z)/\partial z = z-y)}_{⑧}$$

From ⑦: $k_1(y,z) = \int(-z)\,dy + k_2(z) = -yz + k_2(z)$ ⑨

From ⑨: $\partial k_1(y,z)/\partial z = -y + dk_2(z)/dz$ ⑩

Equate ⑧ with ⑩:

$dk_2(z)/dz = z \Rightarrow k_2(z) = \int z\,dz + C = z^2/2 \quad (C\in\mathbb{R})$ ⑪

Back-substitute ⑪ in ⑨ and then ⑨ in ④ to get

$k_1(y,z) = -yz + z^2/2 + C$

$\therefore \varphi(x,y,z) = (x^2+z^2)/2 + y(x-z) + C$

Therefore, by the fundamental theorem for conservative fields:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C (\operatorname{grad}\varphi)\cdot d\vec{r} = \varphi(0,-2,3) - \varphi(1,0,1)\\ &= [(0^2+3^2)/2 + (-2)(0-3) + C] - [(1^2+1^2)/2 + 0 + C]\\ &= 9/2 + 6 - (1)\\ &= 19/2\end{aligned}$$

$\therefore$ the work done by $\vec{F}$ is $\boxed{19/2 \text{ units of energy}}$

(Joules if $\vec{F}$ is in Newtons and distances are in meters)

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Problem 11, Section 15.4 (Adams)

Question: a) Determine the values of A and B for which
$\vec{F} = (Ax \cdot \ln z)\hat{\imath} + (By^2 z)\hat{\jmath} + (x^2/z + y^3)\hat{k}$ is conservative

b) Find $\int_C [(2x \ln z)dx + (2y^2 z)dy + (y^3)dz]$
where C is the straight line from (1,1,1) to (2,1,2)

Solution:

Part A

$\vec{F}$ can only be conservative if curl $\vec{F} = \vec{0}$; now

$$\text{curl } \vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (Ax \cdot \ln z) & (By^2 z) & (x^2/z + y^3) \end{bmatrix}$$

$$= (3y^2 - By^2)\hat{\imath} - (2x/z - Ax/z)\hat{\jmath} + (0-0)\hat{k}$$

Thus curl $\vec{F} = \vec{0} \Leftrightarrow (3y^2 - By^2 = 0) \wedge (2x/z - Ax/z = 0) \Leftrightarrow (A=2) \wedge (B=3)$

So $\vec{F}$ cannot be conservative for any values of A and B except $A=2, B=3$.

This does not mean $\vec{F}$ actually is conservative for $A=2, B=3$!

(Remember, curl $\vec{F} = \vec{0}$ is a necessary condition for $\vec{F}$ to be conservative, but not a sufficient one.)

To find out if $\vec{F}$ is conservative for $A=2, B=3$, try to find a potential for $\vec{F}$. Let $\vec{F} = \text{grad } \varphi(x,y,z)$; then

$(\partial\varphi/\partial x = 2x \ln z)_{①} \wedge (\partial\varphi/\partial y = 3y^2 z)_{②} \wedge (\partial\varphi/\partial z = x^2/z + y^3)_{③}$

From ②: $\varphi = \int 3y^2 z \, dy + k_1(x,z) = y^3 z + k_1(x,z)$ ④

From ④: $\partial\varphi/\partial x = \partial k_1(x,z)/\partial x$ ⑤

$\partial\varphi/\partial z = y^3 + \partial k_1(x,z)/\partial z$ ⑥

Equate ① with ⑤ and ③ with ⑥:

$(\partial k_1(x,z)/\partial x = 2x \ln z)_{⑦} \wedge (\partial k_1(x,z)/\partial z = x^2/z)_{⑧}$

From ⑧: $k_1(x,z) = \int (x^2/z)\, dz + k_2(x) = x^2 \ln z + k_2(x)$ ⑨

From ⑨: $\partial k_1(x,z)/\partial x = 2x \ln z + dk_2(x)/dx$ ⑩

Equate ⑦ with ⑩:

$dk_2(x)/dx = 0 \Rightarrow k_2(x) = C \quad (C \in \mathbb{R})$ ⑪

Back-substitute (11) in (9) and then (9) in (4) to get

$K_1(x,z) = x^2 \ln z + C$

$\varphi(x,y,z) = x^2 \ln z + y^3 z + C$

So a potential function does exist for $\vec{F}$ with $A=2, B=3$, and thus $\vec{F}$ is conservative for these values of $A$ and $B$

$\therefore$ $\boxed{\vec{F} \text{ is conservative for } A=2 \text{ and } B=3 \text{ and is not conservative otherwise}}$

Part B

We could evaluate the whole line integral normally, but we can save some work by decomposing the integral into conservative and non-conservative parts:

$$\int_C [(2x \ln z)\,dx + (2y^2 z)\,dy + (y^3)\,dz]$$
$$= \int_C [(2x \ln z)\hat{\imath} + (3y^2 z)\hat{\jmath} + (x^2/z + y^3)\hat{k}]\cdot d\vec{r} - \int_C [(y^2 z)\hat{\jmath} + (x^2/z)\hat{k}]\cdot d\vec{r}$$

The first integrand is $\vec{F}$ with $A=2$ and $B=3$, and so by the fundamental theorem for conservative fields

$$\int_C [(2x\ln z)\hat{\imath} + (3y^2z)\hat{\jmath} + (x^2/z + y^3)\hat{k}]\cdot d\vec{r} = \varphi(2,1,2) - \varphi(1,1,1)$$
$$= [2^2 \ln 2 + 1^3 \cdot 2 + C] - [1^2 \ln 1 + 1^3 \cdot 1 + C]$$
$$= 4 \ln 2 + 2 - 1 = 4 \ln 2 + 1$$

We still have to evaluate the (simpler) second line integral the normal way:

1) Write the integral in standard scalar or standard vector form

Let $\vec{G} = (y^2 z)\hat{\jmath} + (x^2/z)\hat{k}$ ; then the integral is $\int_C \vec{G}\cdot d\vec{r}$, which is in standard vector form

2) Parametrize $C$: $C \sim \vec{r}(t) = t\hat{\imath} + \hat{\jmath} + t\hat{k}$ , $t$ from 1 to 2

3) Integrand: $\vec{G} = (1^2 \cdot t)\hat{\jmath} + (t^2/t)\hat{k} = t\hat{\jmath} + t\hat{k}$

4) $d\vec{r} = (d\vec{r}(t)/dt)\,dt = (\hat{\imath} + \hat{k})\,dt$

5) Evaluate the integral:

$$\int_C \vec{G}\cdot d\vec{r} = \int_1^2 (t\hat{\jmath} + t\hat{k})\cdot(\hat{\imath} + \hat{k})\,dt = \int_1^2 t\,dt = \left[t^2/2\right]_1^2$$
$$= (1/2)(2^2 - 1^2) = 3/2$$

$$\therefore \int_C [(2x \ln z)\,dx + (2y^2 z)\,dy + (y^3)\,dz] = 4 \ln 2 + 1 - 3/2 = \boxed{4 \ln 2 - 1/2}$$

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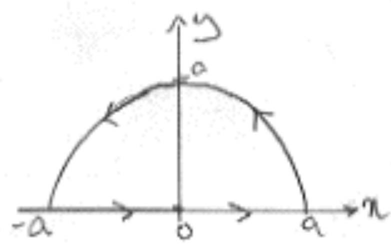
## Problem 1, Section 16.3 (Adams)

Question: Evaluate $\oint_C [(\sin x + 3y^2)\,dx + (2x - e^{-y^2})\,dy]$
where $C$ is the boundary of the half-disk $x^2 + y^2 \le a^2$, $y \ge 0$, oriented counterclockwise (CCW)

Solution:

First rewrite the integral as $\oint_C \vec{F} \cdot d\vec{r}$, where
$\vec{F} = (\sin x + 3y^2)\hat{\imath} + (2x - e^{-y^2})\hat{\jmath}$




Let $S$ be the half disk bounded by $C$;
then by Stokes' theorem,
$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$
where, by the right-hand rule described in the slides, $d\vec{S}$ must point in the $+\hat{k}$ direction if $C$ is to describe the positively-oriented boundary of $S$ (i.e. $d\vec{S} = +\hat{k}\,dS$)

Now evaluate the surface integral
1) Sketch the region: see above
2) Pick a coordinate system: plane polar (we have a disk)
3) Parametrize the surface: we already have only two parameters, $r$ and $\theta$; from the sketch:

$\vec{r}_S(r, \theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$, $(0 \le r \le a) \wedge (0 \le \theta \le \pi)$

4) Integrand:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\sin x + 3y^2) & (2x - e^{-y^2}) & 0 \end{vmatrix} = (2-6y)\,\hat{k}$$

In plane-polar coordinates,

$$\text{curl}\,\vec{F} = (2 - 6r\sin\theta)\,\hat{k}$$

5) Differential surface element: $dS = r\,dr\,d\theta$ plane polar Jacobian determinant

$$\Rightarrow d\vec{S} = (r\hat{k})\,dr\,d\theta \quad (\text{since } d\vec{S} = \hat{k}\,dS)$$

6) Evaluate the integral

$$\iint_S (\text{curl}\,\vec{F})\cdot d\vec{S} = \int_0^a dr \int_0^\pi d\theta\,(2 - 6r\sin\theta)(r)$$
$$= 2\int_0^a r\,dr \int_0^\pi d\theta - 6\int_0^a r^2\,dr \int_0^\pi \sin\theta\,d\theta$$
$$= 2\left[r^2/2\right]_0^a (\pi - 0) - 6\left[r^3/3\right]_0^a \cdot \left[-\cos\theta\right]_0^\pi$$
$$= \pi a^2 - 4a^3$$

$$\therefore \oint_C \left[(\sin x + 3y^2)\,dx + (2x - e^{-y^2})\,dy\right] = \boxed{\pi a^2 - 4a^3}$$