MATH 264 (Advanced Calculus), Fall 2006, McGill University
Thursday Tutorial, 11:35 A.M. in Wilson 103
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TUTORIAL 6 SOLUTIONS (OCTOBER 19, 2006) – VERSION 1

Problems 5 and 7, Section 15.1 (Adams)

Question: Sketch the given plane vector field and determine its field lines:

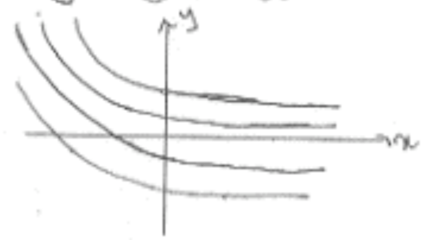
5) $\vec{F}(x,y) = e^x\hat{\imath} + e^{-x}\hat{\jmath}$

7) $\vec{F}(x,y) = \nabla[\ln(x^2+y^2)]$

Solution:

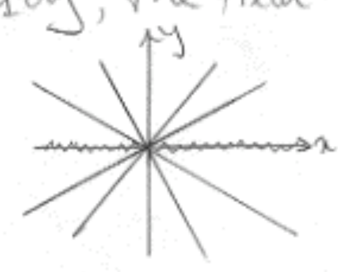
We will leave the sketching of the fields to Maple.
The field lines may be found using the equation
$dx/F_1 = dy/F_2 \iff F_2\,dx = F_1\,dy$

5) $F_2\,dx = F_1\,dy \Rightarrow e^{-x}\,dx = e^x\,dy \Rightarrow dy = e^{-2x}\,dx$
$\Rightarrow \int dy = \int e^{-2x}\,dx + C \quad (C \in \mathbb{R})$
$\Rightarrow y = (-1/2)\,e^{-2x} + C$
This is a family of exponentials (sketched on the right)



7) $\vec{F}(x,y) = \text{grad}(\ln(x^2+y^2)) = \dfrac{\partial(\ln(x^2+y^2))}{\partial x}\hat{\imath} + \dfrac{\partial(\ln(x^2+y^2))}{\partial y}\hat{\jmath}$
$= \left(\dfrac{1}{x^2+y^2}\right)\cdot 2x\cdot\hat{\imath} + \left(\dfrac{1}{x^2+y^2}\right)\cdot 2y\cdot\hat{\jmath}$

So $F_2\,dx = F_1\,dy \Rightarrow \left(\dfrac{2y}{x^2+y^2}\right)dx = \left(\dfrac{2x}{x^2+y^2}\right)dy$
$\Rightarrow (1/y)\,dy = (1/x)\,dx \Rightarrow \int(1/y)\,dy = \int(1/x)\,dx + C \quad (C \in \mathbb{R})$
$\Rightarrow \ln y = \ln x + C \Rightarrow y = e^c e^{\ln x} = e^c x$
$\Rightarrow y = Cx \quad (C \in \mathbb{R} \wedge C \neq 0)$
Since $y = 0$ also satisfies $F_2\,dx = F_1\,dy$, the field the field lines are given by $y = Cx$ $(C \in \mathbb{R})$, which is a family of lines through the origin except for $x = 0$ (sketched on the right)



Example 1 (p.958), Section 16.2 (Adams)

Question: Show that $\vec{F} = (x^2+yz)\hat{\imath} - 2y(x+z)\hat{\jmath} + (xy+z^2)\hat{k}$ is solenoidal and find a vector potential for it

Solution:

$\operatorname{div}\vec{F} = \partial F_1/\partial x + \partial F_2/\partial y + \partial F_3/\partial z = 2x - 2(x+z) + 2z = 0 \Rightarrow \boxed{\vec{F} \text{ is solenoidal}}$

Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ be a vector potential for $\vec{F}$

Then $\vec{F} = \operatorname{curl}\vec{A} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{bmatrix}$

$= (\partial A_3/\partial y - \partial A_2/\partial z)\hat{\imath} + (\partial A_1/\partial z - \partial A_3/\partial x)\hat{\jmath} + (\partial A_2/\partial x - \partial A_1/\partial y)\hat{k}$

Equating the components of $\vec{F}$ with those of $\operatorname{curl}(\vec{A})$:

$$\begin{cases} \partial A_3/\partial y = \partial A_2/\partial z + x^2+yz & (1) \\ \partial A_1/\partial z = \partial A_3/\partial x - 2y(x+z) & (2) \\ \partial A_2/\partial x = \partial A_1/\partial y + xy+z^2 & (3) \end{cases}$$

We're trying to solve for A_1, A_2, and A_3; each partial derivative on one of these terms represents one "degree of freedom" (d.f.) in our equations.

There are 3 partials (in x, y, and z) for the three components (A_1, A_2, and A_3), and so we have 9 d.f. to start with.

The three equations above (from the constraint $\vec{F} = \operatorname{curl}(\vec{A})$) each remove one d.f., so we're left with 6 d.f. that we can play with.

Let's set $A_2 = 0$ (this is an arbitrary choice)

This sets $\partial A_2/\partial x = \partial A_2/\partial y = \partial A_2/\partial z = 0$, so we've used up 3 d.f. and have 3 left.

So our equations become

$$\begin{cases} \partial A_3/\partial y = x^2+yz & (1) \\ \partial A_1/\partial z = \partial A_3/\partial x - 2xy - 2yz & (2) \\ \partial A_1/\partial y = -xy - z^2 & (3) \end{cases}$$

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From 1: $A_3 = \int (x^2 + yz)\, dy + k_1(x,z)$
$= x^2y + y^2z/2 + k_1(x,z)$ 4

We can set arbitrary function $k_1(x,z) = 0$; this uses 2 d.f. ($\partial k_1(x,z)/\partial x = \partial k_1(x,z)/\partial z = 0$, which in turn constrains $\partial A_3/\partial x$ and $\partial A_3/\partial z$ through Equation 4); we have 1 d.f. left

So: $A_3 = x^2y + y^2z/2$ 4

Then: $\partial A_3/\partial x = \partial/\partial x\,(x^2y + y^2z/2) = 2xy$ 5

Use 5 in 2 to get the system:

$$\begin{cases} \partial A_1/\partial y = -xy - z^2 & \text{3 (from before)} \\ \partial A_1/\partial z = -2yz & \text{6 (use 5 in 2)} \end{cases}$$

From 6: $A_1 = \int (-2yz)\, dz + k_2(x,y) = -yz^2 + k_2(x,y)$ 7

We can't take $k_2(x,y) = 0$ because that would use up 2 d.f. and we only have 1 left; however, from 7:

$\partial A_1/\partial y = -z^2 + \partial k_2(x,y)/\partial y$ 8

Now equate 8 and 6:

$\partial k_2(x,y)/\partial y = -xy$

$\Rightarrow k_2(x,y) = \int (-xy)\, dy + k_3(x) = -xy^2/2 + k_3(x)$ 9

Now we can use our last d.f. to set $k_3(x) = 0$ (hence $dk_3(x)/dx = 0$, which constrains $\partial k_2(x,y)/\partial x$ and also $\partial A_1/\partial x$ via 7)

So $k_2(x,y) = -xy^2/2$; use this in 7 to get:

$A_1 = -yz^2 - xy^2/2$

To summarize:

$$\begin{cases} A_1 = -yz^2 - xy^2/2 & \text{(from 7)} \\ A_2 = 0 & \text{(constraint at the beginning)} \\ A_3 = x^2y + y^2z/2 & \text{(from 4)} \end{cases}$$

$\therefore$ a vector potential for $\vec{F}$ is

$$\boxed{\vec{A} = (-yz^2 - xy^2/2)\,\hat{\imath} + (x^2y + y^2z/2)\,\hat{k}}$$

Note that this is not the only possible answer; any function satisfying curl $\vec{A} = \vec{F}$ will work also.

It's always a good idea to check that curl $\vec{A} = \vec{F}$ for the solution found. In this case:

$$\text{curl}\,\vec{A} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (-yz^2 - xy^2/2) & 0 & (x^2y + y^2z/2) \end{bmatrix}$$

$$= \hat{\imath}\left[x^2 + yz\right] - \hat{\jmath}\left[2xy - (-2yz)\right] + \hat{k}\left[0 - (-z^2 - xy)\right]$$

$$= (x^2 + yz)\,\hat{\imath} - 2y(x+z)\,\hat{\jmath} + (xy + z^2)\,\hat{k}$$

$$= \vec{F}$$

as required

Problem 9, Section 15.2 (Adams)

Question: Show that $\vec{F}(x,y,z) = (2x/z)\hat{\imath} + (2y/z)\hat{\jmath} - [(x^2+y^2)/z^2]\hat{k}$ is a conservative vector field, and find a potential function for it. Describe the equipotential surfaces and find the field lines of $\vec{F}$.

Solution:

The way to prove a field is conservative is to find a potential for it.
Let $\vec{F} = \text{grad}\ \varphi(x,y,z)$; then

$(\partial\varphi/\partial x = 2x/z)$ ← ① $\wedge$ $(\partial\varphi/\partial y = 2y/z)$ ← ② $\wedge$ $(\partial\varphi/\partial z = -(x^2+y^2)/z^2)$ ← ③

From ①: $\varphi = \int (2x/z)\,dx + k_1(y,z) = x^2/z + k_1(y,z)$ ④

From ④: $(\partial\varphi/\partial y = \partial k_1(y,z)/\partial y)$ ← ⑤ $\wedge$ $(\partial\varphi/\partial z = -x^2/z^2 + \partial k_1(y,z)/\partial z)$ ← ⑥

Equate ② with ⑤ and ③ with ⑥:

$(\partial k_1(y,z)/\partial y = 2y/z)$ ← ⑦ $\wedge$ $(\partial k_1(y,z)/\partial z = -y^2/z^2)$ ← ⑧

From ⑦: $k_1(y,z) = \int (2y/z)\,dy + k_2(z) = y^2/z + k_2(z)$ ⑨

From ⑨: $\partial k_1(y,z)/\partial z = -y^2/z^2 + dk_2(z)/dz$ ⑩

Equate ⑧ with ⑩:

$dk_2(z)/dz = 0 \Rightarrow k_2(z) = \int 0\,dz + C = C \quad (C \in \mathbb{R})$ ⑪

Back-substitute ⑪ in ⑨ and then ⑨ in ④ to get

$k_1(y,z) = y^2/z + C \Rightarrow \varphi(x,y,z) = (x^2+y^2)/z + C$

$\therefore$ $\boxed{\vec{F} \text{ is conservative and a potential for it is } \varphi(x,y,z) = (x^2+y^2)/z + C \text{ where } C \in \mathbb{R}}$

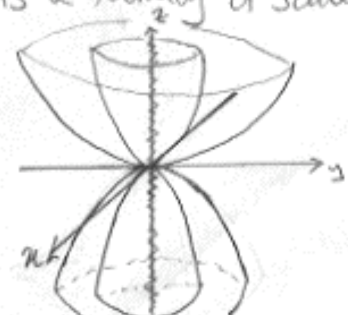
The equipotentials are given by

$\varphi(x,y,z) = C_* \Rightarrow (x^2+y^2)/z = C_* \quad (C_* \in \mathbb{R}$, C absorbed into $C_*)$

If $C_* = 0$, $x^2+y^2 = 0 \Rightarrow (x,y) = (0,0)$, which is the z axis

If $C_* \neq 0$, $z = (1/C_*)(x^2+y^2)$ which is a family of scaled paraboloids

The equipotentials are sketched at the right



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Lastly, the field lines of $\vec{F}$ are given by

$$dx/F_1 = dy/F_2 = dy/F_3 \iff (dx/F_1 = dy/F_2) \wedge (dx/F_1 = dz/F_3)$$

Now $dx/F_1 = dy/F_2 \Rightarrow F_2\,dx = F_1\,dy \Rightarrow (2y/z)\,dx = (2x/z)\,dy$

$$\Rightarrow (1/y)\,dy = (1/x)\,dx \Rightarrow \int (1/y)\,dy = \int (1/x)\,dx$$

$$\Rightarrow \ln y = \ln x + C_1 \quad (C_1 \in \mathbb{R})$$

$$\Rightarrow y = e^{C_1} e^{\ln x} = C_1 x \quad (C_1 \in \mathbb{R} \wedge C_1 \neq 0)$$

Since $y=0$ also solves the equation, we get $y = C_1 x$, $C_1 \in \mathbb{R}$

Also $dx/F_1 = dz/F_3 \Rightarrow F_3\,dx = F_1\,dz$

$$\Rightarrow [-(x^2+y^2)/z^2]\,dx = (2x/z)\,dz$$

$$\Rightarrow [-(x^2 + C_1^2 x^2)/z^2]\,dx = (2x/z)\,dz \quad (\text{since } y = C_1 x)$$

$$\Rightarrow [(x^2 + C_1^2 x^2)/x]\,dx = (-2z^2/z)\,dz$$

$$\Rightarrow (1 + C_1^2)\,x\,dx = -2z\,dz$$

$$\Rightarrow \int (1 + C_1^2)\,x\,dx = \int (-2z)\,dz$$

$$\Rightarrow (1 + C_1^2)(x^2/2) = -z^2 + C_2 \quad (C_2 \in \mathbb{R})$$

$$\Rightarrow x^2 + (C_1 x)^2 = -2z^2 + 2C_2$$

$$\Rightarrow x^2 + y^2 + 2z^2 = C_2 \quad (\text{use } y = C_1 x)$$

$$\Rightarrow x^2 + y^2 + 2z^2 = C_2^2 \quad (\text{since } x^2+y^2+2z^2 \geq 0 \text{ anyway})$$

$\therefore$ the field lines are described by

$$\boxed{\begin{cases} y = C_1 x, & C_1 \in \mathbb{R} \quad (\text{vertical planes through the origin}) \\ x^2+y^2+2z^2 = C_2^2, & C_2 \in \mathbb{R} \quad (\text{ellipsoids centered at the origin}) \end{cases}}$$

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Problem 3 (Section 15.2, Adams)

Question: Is $\vec{F}(x,y) = \frac{x}{x^2+y^2}\hat{\imath} - \frac{y}{x^2+y^2}\hat{\jmath}$ conservative? If so, find a potential function for it.

Solution:

Assume $\vec{F}(x,y)$ is conservative

Then $\underbrace{\exists}_{\text{"there exists"}} \varphi(x,y) \underbrace{\text{s.t.}}_{\text{"such that"}} \vec{F}(x,y) = \nabla\varphi(x,y)$

Then $\underbrace{(\partial\varphi/\partial x = x/x^2+y^2)}_{①} \wedge \underbrace{(\partial\varphi/\partial y = -y/x^2+y^2)}_{②}$

From equation ①:

$\varphi = \int \frac{x}{x^2+y^2}dx + k(y) = \left(\frac{1}{2}\right)\int \frac{du}{u} + k(y)$ | Let $u = x^2+y^2$, $du = 2x\,dx \Rightarrow x\,dx = du/2$

$= (1/2)\ln u + k(y) = (1/2)\ln(x^2+y^2) + k(y)$

Hence $\partial\varphi/\partial y = (1/2)\left[(1/(x^2+y^2))\times 2y\right] + \underbrace{k'(y)}_{\text{unknown function of } \underline{y \text{ only}}} = y/(x^2+y^2) + k'(y)$

From equation ②:

$y/(x^2+y^2) + k'(y) = -y/(x^2+y^2)$

$\Rightarrow k'(y) = -2y/(x^2+y^2)$

$\Rightarrow$ contradiction! ($k'(y)$ cannot be a function of x, only of y)

So there is no function φ satisfying $\vec{F} = \nabla\varphi$, and hence $\vec{F}$ is not conservative.