**TUTORIAL 4 SOLUTIONS (OCTOBER 5, 2006) – VERSION 1**

# Problems 5, 7, and 9, Section 16.1 (Adams)

Question: Find div $\vec{F}$ and curl $\vec{F}$ for:

5) $\vec{F} = x\hat{\imath} + x\hat{k}$

7) $\vec{F} = f(x)\hat{\imath} + g(y)\hat{\jmath} + h(z)\hat{k}$

9) $\vec{F}(r,\theta) = r\hat{\imath} + (\sin\theta)\hat{\jmath}$

Note that in Problem 9, $\vec{F}$ is expressed using polar coordinates

Solution:

For $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$, $\operatorname{div}\vec{F} = \partial F_1/\partial x + \partial F_2/\partial y + \partial F_3/\partial z$

and $\operatorname{curl}\vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{bmatrix}$

5) $\operatorname{div}\vec{F} = \partial x/\partial x + 0 + \partial x/\partial z = \boxed{1}$

$\operatorname{curl}\vec{F} = \hat{\imath}\left[\partial x/\partial y - 0\right] - \hat{\jmath}\left[\partial x/\partial x - \partial x/\partial z\right] + \hat{k}\left[0 - \partial x/\partial y\right] = \boxed{-\hat{\jmath}}$

7) $\operatorname{div}\vec{F} = \boxed{f'(x) + g'(y) + h'(z)}$

$\operatorname{curl}\vec{F} = \hat{\imath}\left[\partial h/\partial y - \partial g/\partial z\right] - \hat{\jmath}\left[\partial h/\partial x - \partial f/\partial z\right] + \hat{k}\left[\partial g/\partial x - \partial f/\partial y\right] = \boxed{\vec{0}}$

Vector $\vec{0}$, not scalar zero!

9) div and curl are defined in rectangular coordinates, so convert $\vec{F}$ to rectangular using $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{cases}$

$\vec{F} = \sqrt{x^2+y^2}\,\hat{\imath} + \sin(\arctan(y/x))\,\hat{\jmath}$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\sin(\arctan(y/x))\right) + 0$

$= \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + \cos(\arctan(y/x)) \cdot \frac{1}{1+(y/x)^2} \cdot (1/x)$

Now reconvert to polar using $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ and $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{cases}$

$$\operatorname{div}\vec{F} = \frac{r\cos\theta}{r} + \cos(\theta)\cdot\frac{1}{1+\tan^2\theta}\cdot\frac{1}{r\cos\theta}$$
$$= \cos\theta + (1/r)(1/\sec^2\theta) = \cos\theta + (1/r)\cos^2\theta$$
$$= \boxed{(\cos\theta)[1 + (1/r)\cos\theta]}$$

$$\operatorname{curl}\vec{F} = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sqrt{x^2+y^2} & \sin(\arctan(y/x)) & 0 \end{bmatrix}$$
$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}\left[\frac{\partial}{\partial x}(\sin(\arctan(y/x))) - \frac{\partial}{\partial y}(\sqrt{x^2+y^2})\right]$$
$$= \hat{k}\left[\cos(\arctan(y/x))\cdot\frac{1}{1+(y/x)^2}\cdot y(-x^{-2}) - \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x\right]$$
$$= \hat{k}\left[\cos(\theta)\cdot\frac{1}{1+\tan^2\theta}\cdot r\sin\theta\cdot\frac{-1}{r^2\cos^2\theta} - \frac{x}{r}\right]$$
$$= \hat{k}\left[\frac{-r\sin\theta\cos\theta}{\sec^2\theta\cdot r^2\cos^2\theta} - \frac{r\cos\theta}{r}\right]$$
$$= \hat{k}\left[-(1/r)\cdot\sin\theta\cos\theta - \cos\theta\right]$$
$$= \boxed{[(-\cos\theta)(1 + (1/r)\sin\theta)]\hat{k}}$$

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Problem 3, Section 16.2 (Adams)

Question: Prove that $\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\nabla\times\vec{G})$

Solution:

Proving vector identities is basically very boring: expand the left- and right-hand sides (LHS and RHS) and show that they're equal.

LHS

$$\vec{F}\times\vec{G} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ F_1 & F_2 & F_3\\ G_1 & G_2 & G_3\end{bmatrix} = \hat{\imath}\left[F_2G_3 - F_3G_2\right] + \hat{\jmath}\left[F_3G_1 - F_1G_3\right] + \hat{k}\left[F_1G_2 - F_2G_1\right]$$

$$\begin{aligned}\text{So LHS} = \operatorname{div}(\vec{F}\times\vec{G}) &= (\partial F_2/\partial x)\cdot G_3 + F_2\cdot(\partial G_3/\partial x) - (\partial F_3/\partial x)G_2 - F_3(\partial G_2/\partial x)\\ &+ (\partial F_3/\partial y)\cdot G_1 + F_3\cdot(\partial G_1/\partial y) - (\partial F_1/\partial y)\cdot G_3 - F_1(\partial G_3/\partial y)\\ &+ (\partial F_1/\partial z)\cdot G_2 + F_1\cdot(\partial G_2/\partial z) - (\partial F_2/\partial z)\cdot G_1 - F_2\cdot(\partial G_1/\partial z)\end{aligned}$$

RHS

$$\nabla\times\vec{F} = \hat{\imath}\left[\partial F_3/\partial y - \partial F_2/\partial z\right] + \hat{\jmath}\left[\partial F_1/\partial z - \partial F_3/\partial x\right] + \hat{k}\left[\partial F_2/\partial x - \partial F_1/\partial y\right]$$

$$(\nabla\times\vec{F})\cdot\vec{G} = \left[\partial F_3/\partial y - \partial F_2/\partial z\right]G_1 + \left[\partial F_1/\partial z - \partial F_3/\partial x\right]G_2 + \left[\partial F_2/\partial x - \partial F_1/\partial y\right]G_3$$

By symmetry,

$$\vec{F}\cdot(\nabla\times\vec{G}) = (\nabla\times\vec{G})\cdot\vec{F} = \left[\partial G_3/\partial y - \partial G_2/\partial z\right]F_1 + \left[\partial G_1/\partial z - \partial G_3/\partial x\right]F_2 + \left[\partial G_2/\partial x - \partial G_1/\partial y\right]F_3$$

$$\begin{aligned}\therefore \text{RHS} &= (\nabla\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\nabla\times\vec{G})\\ &= (\partial F_3/\partial y)\cdot G_1 - (\partial F_2/\partial y)\cdot G_1 + (\partial F_1/\partial z)\cdot G_2 - (\partial F_3/\partial x)\cdot G_2\\ &+ (\partial F_2/\partial z)\cdot G_3 - (\partial F_1/\partial y)\cdot G_3 - F_1(\partial G_3/\partial y) + F_1(\partial G_2/\partial z)\\ &- F_2(\partial G_1/\partial z) + F_2(\partial G_3/\partial x) - F_3(\partial G_2/\partial x) + F_3(\partial G_1/\partial y)\end{aligned}$$

Comparing terms, we see that LHS = RHS. QED

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## Problem 13, Section 15.6 (Adams)

Question: Find the flux of $\vec{F} = m\vec{r}/|\vec{r}|^3$ out of the surface of the cube $-a \le x, y, z \le a$ $(a>0)$

Solution:
This is a straightforward application of Gauss's law, which is stated in the Tutorial 4 slides.
The flux is

$$\begin{aligned}
\iint_{cube} \vec{F}\cdot d\vec{S} &= \iint_{cube} \left(m\vec{r}/|\vec{r}|^3\right)\cdot d\vec{S} \\
&= m\left[\iint_{cube} \left(\vec{r}/|\vec{r}|^3\right)\cdot d\vec{S}\right] \\
&= m(+4\pi) \qquad \text{(by Gauss's law)} \\
&= 4\pi m
\end{aligned}$$

$\therefore$ the flux is $\boxed{4\pi m}$

## Problem #1, Section 15.6 (Adams)

Question: Find the flux of $\vec{F} = x\hat{\imath} + z\hat{\jmath}$ out of the tetrahedron bounded by the coordinate planes and the plane $x+2y+3z=6$

Solution:

We'll do four integrals, one over each face of the tetrahedron

1) Sketch the surfaces

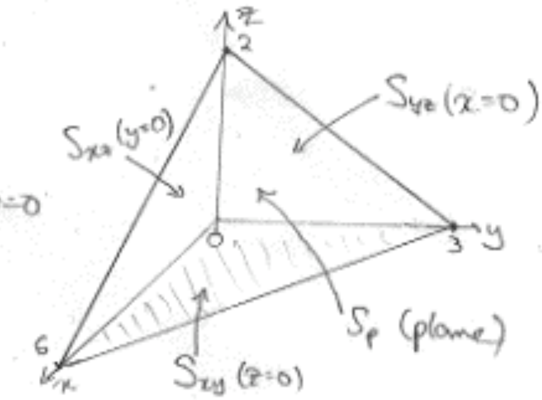
We know the region lies in the first octant $(x,y,z \geq 0)$

We'll find the lines of intersection of the plane with the three planes $x=0, y=0$, and $z=0$

$x=0$ (yz plane): $2y+3z=6 \Rightarrow z=(-\frac{2}{3})y+2$

$y=0$ (xz-plane): $x+3z=6 \Rightarrow z=(-\frac{1}{3})x+2$

$z=0$ (xy-plane): $x+2y=6 \Rightarrow y=(-\frac{1}{2})x+3$




2) Pick a coordinate system.

For all four faces of the tetrahedron, Cartesian will do nicely (although we'll only need two of the three variables $x,y,z$ for each face of the shape)

3) Parametrize the surface

We have four surfaces to parametrize: the three faces of the tetrahedron lying in the coordinate planes (denoted $S_{xy}, S_{xz}$, and $S_{yz}$), plus the fourth face described by $x+2y+3z=6$ (denoted $S_p$)

$S_{xy}$: $\vec{r}_{S_{xy}}(x,y) = x\hat{\imath} + y\hat{\jmath}$ , $(0 \leq x \leq 6) \wedge (0 \leq y \leq (-\frac{1}{2})x+3)$

$S_{xz}$: $\vec{r}_{S_{xz}}(x,z) = x\hat{\imath} + z\hat{k}$ , $(0 \leq x \leq 6) \wedge (0 \leq z \leq (-\frac{1}{3})x+2)$

$S_{yz}$: $\vec{r}_{S_{yz}}(y,z) = y\hat{\jmath} + z\hat{k}$ , $(0 \leq y \leq 3) \wedge (0 \leq z \leq (-\frac{2}{3})y+2)$

To parametrize $S_p$, we note that $x+2y+3z=6$ on this surface; let's (arbitrarily) solve for $x$ to get $x=6-2y-3z$

Hence we've parametrized $S_p$ in terms of $y$ and $z$; we now need to find bounds for $y$ and $z$. To do this, project $S_p$ onto the $yz$ plane, which in fact results in the surface $S_{yz}$. Thus:

$S_p$: $\vec{r}_{S_p}(y,z) = (6-2y-3z)\hat{\imath} + y\hat{\jmath} + z\hat{k}$ , $(0 \leq y \leq 3) \wedge (0 \leq z \leq (-\frac{2}{3})y+2)$

4) Write the integrand (in terms of the parameters).

On $S_{xy}$, $\vec{F} = x\hat{\imath}$ (since $z=0$) On $S_{xz}$, $\vec{F} = x\hat{\imath} + z\hat{\jmath}$

On $S_{yz}$, $\vec{F} = z\hat{\jmath}$ (since $x=0$)

On $S_P$, $\vec{F} = (6-2y-3z)\hat{\imath} + z\hat{\jmath}$ (since $x = 6-2y-3z$)

5) Find the differential surface elements $d\vec{S}$

For $S_{xy}$, $S_{xz}$, and $S_{yz}$, we can easily find the unit normals:

$\hat{N}_{xy} = -\hat{k}$ , $\hat{N}_{xz} = -\hat{\jmath}$ , $\hat{N}_{yz} = -\hat{\imath}$

(note the sign; the normals must point outward, because the question asks for outward flux; see the sketch)

Noting that $S_{xy}$, $S_{xz}$, and $S_{yz}$ are described by $z=0$, $y=0$, and $x=0$ respectively, and are thus vertical planes with no change in $z$, $y$, and $x$ respectively: $dS_{xy} = dx\,dy$, $dS_{xz} = dx\,dz$, $dS_{yz} = dy\,dz$

$\therefore d\vec{S}_{xy} = \hat{N}_{xy}\,dS_{xy} = (-\hat{k})\,dx\,dy$, $d\vec{S}_{xz} = (-\hat{\jmath})\,dx\,dz$, and $d\vec{S}_{yz} = (-\hat{\imath})\,dy\,dz$

For the surface $S_P$:

$\partial\vec{r}_{SP}/\partial y = -2\hat{\imath} + \hat{\jmath}$ and $\partial\vec{r}_{SP}/\partial z = -3\hat{\imath} + \hat{k}$

$$\therefore d\vec{S}_P = \pm\left(\frac{\partial\vec{r}_{SP}}{\partial y} \times \frac{\partial\vec{r}_{SP}}{\partial z}\right) dy\,dz = \pm\left(\det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\right) dy\,dz$$

$$= \pm\left[(1)\hat{\imath} - (-2)\hat{\jmath} + (3)\hat{k}\right] dy\,dz$$

$$= \pm\left[\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right] dy\,dz$$

We want $d\vec{S}_P$ to point outward; from the sketch we see that this means $d\vec{S}_P$ must have a positive $\hat{k}$ component.

$$\therefore d\vec{S}_P = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\,dy\,dz$$

6) Evaluate the integral by iteration

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_{xy}} \vec{F}\cdot d\vec{S}_{xy} + \iint_{S_{xz}} \vec{F}\cdot d\vec{S}_{xz} + \iint_{S_{yz}} \vec{F}\cdot d\vec{S}_{yz} + \iint_{S_P} \vec{F}\cdot d\vec{S}_P$$

Now:

$$\iint_{S_{xy}} \vec{F}\cdot d\vec{S}_{xy} = \int_0^6 dx \int_0^{(-1/2)x+3} dy\,\left[(x\hat{\imath})\cdot(-\hat{k})\right] = 0$$ ← could have predicted this: $\vec{F}$ has no component normal to $S_{xy}$

$$\iint_{S_{xz}} \vec{F}\cdot d\vec{S}_{xz} = \int_0^6 dx \int_0^{(-1/3)x+2} dz\,\left[(x\hat{\imath} + z\hat{\jmath})\cdot(-\hat{\jmath})\right]$$

$$= -\int_0^6 dx \int_0^{(-1/3)x+2} z\,dz = -\int_0^6 dx \left[z^2/2\right]_0^{(-1/3)x+2}$$

$$= -(1/2)\int_0^6 \left[(1/9)x^2 - (4/3)x + 4\right] dx$$

$$= (-1/18)\int_0^6 x^2\,dx + (2/3)\int_0^6 x\,dx - 2\int_0^6 dx$$

$$= -4$$

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$$\iint_{S_{yz}} \vec{F}\cdot d\vec{S}_{yz} = \int_0^3 dy \int_0^{(-2/3)y+2} dz \left[(z\hat{j})\cdot(-\hat{i})\right] = 0$$ ← $x=0$ on $S_{yz}$, so the normal component of $\vec{F}$ to $S_{yz}$ disappears

$$\iint_{S_P} \vec{F}\cdot d\vec{S}_P = \int_0^3 dy \int_0^{(-2/3)y+2} dz \left[((6-2y-3z)\hat{i} + z\hat{j})\cdot(\hat{i}+2\hat{j}+3\hat{k})\right]$$
$$= \int_0^3 dy \int_0^{(-2/3)y+2} dz \left[6-2y-3z+2z\right]$$
$$= \int_0^3 dy \left[6z - 2yz - z^2/2\right]_0^{(-2/3)y+2}$$
$$= \int_0^3 \left[(6-2y)(2-(2/3)y) - (1/2)\left[(-2/3)y+2\right]^2\right] dy$$
$$= \int_0^3 \left[12 - 4y - 4y + (4/3)y^2 - (1/2)\left[(4/9)y^2 - (8/3)y + 4\right]\right] dy$$
$$= \int_0^3 \left[(10/9)y^2 - (20/3)y + 10\right] dy$$
$$= \left[(10/27)y^3 - (10/3)y^2 + 10y\right]_0^3$$
$$= 10$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = 0 + (-4) + 0 + 10 = \boxed{6}$$

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