TUTORIAL 3 SOLUTIONS (SEPTEMBER 28, 2006) – VERSION 1

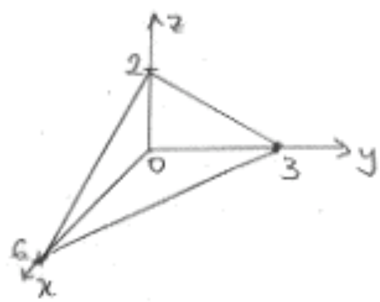
Problem 4, MATH 264 Assignment 1, Fall 2006

Question: With the help of the formula $\hat{N}\,dS = \vec{r}_u \times \vec{r}_v\,du\,dv$ and a suitable parametrization, calculate the surface area of the piece of the plane $x+2y+3z=6$ that lies in the first octant.

Solution:

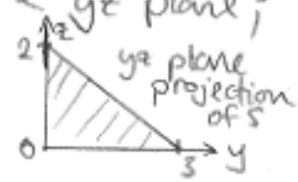
Let S denote the piece of plane described; we want its area, given by $\iint_S dS$. Use the method in Tutorial 3 ($dS = |\hat{N}\,dS|$)

1) Sketch the region: easy



2) Pick a coordinate system:
Start with Cartesian (x,y,z), note that on the plane's surface, $x = 6-2y-3z$
So we use y and z as our coordinate variables and take $x = 6-2y-3z$
(we could of course have solved for y or z also instead of x, as long as there are only two parameters for the surface)

3) Parametrize the surface
To get bounds on y and z, project the surface onto the yz plane; this gives the region shown at the right, for which $0 \le y \le 3$ and $0 \le z \le (-2/3)y+2$



$\therefore\ \vec{r}_S(y,z) = (6-2y-3z)\hat{\imath} + y\hat{\jmath} + z\hat{k}$ where $\begin{cases} 0 \le y \le 3 \\ 0 \le z \le (-2/3)y+2 \end{cases}$

4) Integrand: 1

5) Find dS: $\partial\vec{r}_S/\partial y = -2\hat{\imath}+\hat{\jmath}$ and $\partial\vec{r}_S/\partial z = -3\hat{\imath}+\hat{k}$

$$d\vec{S} = \hat{n}\,dS = \pm\left(\frac{\partial\vec{r}_S(y,z)}{\partial y} \times \frac{\partial\vec{r}_S(y,z)}{\partial z}\right)dy\,dz = \pm\det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} dy\,dz$$

$$= \pm(\hat{\imath}+2\hat{\jmath}+3\hat{k})\,dy\,dz$$

$$dS = |\hat{n}\,dS| = \sqrt{1^2+2^2+3^2}\,dy\,dz = \sqrt{14}\,dy\,dz$$

6) Evaluate the integral

$$\iint_S dS = \int_0^3 dy \int_0^{(-2/3)y+2} dz\,\sqrt{14} = \sqrt{14}\int_0^3 [(-2/3)y+2]\,dy$$

$$= \sqrt{14}\left[-y^2/3 + 2y\right]_0^3 = \sqrt{14}(-9/3+6)$$

$$= 3\sqrt{14}$$

$\therefore$ the area of the surface described is $\boxed{3\sqrt{14}\text{ units}^2}$

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Problem 8 (Section 15.5, Adams)

Question: Find the area of the part of the cone $z^2 = x^2+y^2$ that lies inside the cylinder $x^2+y^2 = 2ay$ $(a>0)$

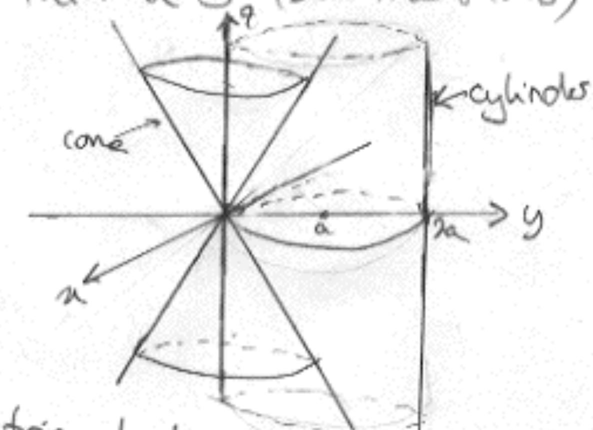
Solution:

Follow the method given in Tutorial 5 (see the slides)

1) Sketch the region

Cylinder: $x^2+y^2 = 2ay$
$\Rightarrow x^2+(y-a)^2 = a^2$
(radius a, center $(0,a)$)

Cone: $z = \pm\sqrt{x^2+y^2}$



We see that the region is symmetric about the xy $(z=0)$ plane, so we'll find the area of the top half only, then multiply by 2 to get total area

2) Choose a coordinate system

Because of the cylinder, cylindrical coordinates is a good choice:
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

However, there are 3 parameters $(r, \theta, \text{and } z)$ in this coordinate system, yet we only need two parameters to describe a surface.

As we will see in the next step, one of the variables will end up being fixed...

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3) Parametrize the surface of integration

Denoting the vector function that describes our surface as $\vec{r}_s$, we have

$$\vec{r}_s(r,\theta,z) = [r\cos\theta]\hat{\imath} + [r\sin\theta]\hat{\jmath} + [z]\hat{k}$$

One of these three parameters will end up being a function of the other two, because we're integrating over a <u>surface</u>.

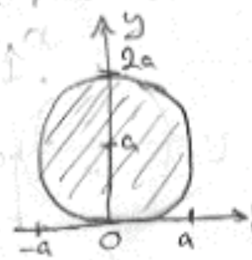
In this case, we're integrating over the cone, so

$$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

(top half only)

Thus z is expressible in terms of r. We need to find bounds for the two remaining parameters, r and θ.

Projecting the surface onto the xy plane gives the filled disk at the right



We see that $0 \le \theta \le \pi$

The equation of the cylinder gives the boundary of the disk:

$$x^2+y^2 = 2ay \Rightarrow r^2 = 2ar\sin\theta \Rightarrow (r = 2a\sin\theta) \vee (r=0)$$

Hence $0 \le r \le 2a\sin\theta$

$\therefore$ our 3D surface may be parametrized as

$$\vec{r}_s(r,\theta) = [r\cos\theta]\hat{\imath} + [r\sin\theta]\hat{\jmath} + r\hat{k}$$

where $0 \le \theta \le \pi$ and $0 \le r \le 2a\sin\theta$

4) Parametrize the integrand

The integrand is just 1 (we want to find area, i.e. $\iint_S dS$).

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5) Find the differential element of surface dS

we will use the formula $dS = \left| \frac{\partial \vec{r}_s}{\partial r} \times \frac{\partial \vec{r}_s}{\partial \theta} \right| dr\, d\theta$

$\partial \vec{r}_s / \partial r = [\cos\theta]\hat{\imath} + [\sin\theta]\hat{\jmath} + \hat{k}$

$\partial \vec{r}_s / \partial \theta = [-r\sin\theta]\hat{\imath} + [r\cos\theta]\hat{\jmath}$

$$\partial \vec{r}_s/\partial r \times \partial \vec{r}_s/\partial \theta = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix}$$

$$= \hat{\imath}[-r\cos\theta] - \hat{\jmath}[+r\sin\theta] + \hat{k}[r\cos^2\theta + r\sin^2\theta]$$

$$= \hat{\imath}[-r\cos\theta] + \hat{\jmath}[r\sin\theta] + \hat{k}[r]$$

$$|\partial \vec{r}_s/\partial r \times \partial \vec{r}_s/\partial \theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{1+1} = \sqrt{2}\, r$$

$\therefore\ dS = \sqrt{2}\, r\, dr\, d\theta$

6) Evaluate the integral

$$\iint_S dS = \int_0^{\pi} d\theta \int_0^{2a\sin\theta} dr\, (\sqrt{2}\, r) = \sqrt{2}\int_0^{\pi} d\theta \left[r^2/2 \right]_0^{2a\sin\theta}$$

$$= (\sqrt{2}/2)\cdot\int_0^{\pi} 4a^2 \sin^2\theta\, d\theta = 2\sqrt{2}\, a^2 \int_0^{\pi} \left[\frac{1-\cos(2\theta)}{2} \right] d\theta$$

$$= \sqrt{2}\, a^2 \left[\int_0^{\pi} d\theta - \int_0^{2\pi} \cos(\theta)\,(d\theta/2) \right]$$

$$= \sqrt{2}\, a^2 [\pi - 0 - 0] = \sqrt{2}\,\pi a^2$$

Remembering that we only considered the top half of the surface, we need to multiply by 2 to get the total surface area.

$\therefore$ the area is $\boxed{2\sqrt{2}\,\pi a^2 \text{ units}^2}$

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Problem 9 (Section 15.5, Adams)

Question: Find the area of the part of the cylinder $x^2+y^2=2ay$ that lies outside the cone $z^2=x^2+y^2$ $(a>0)$ (note the similarities and differences with the previous problem...)

Solution:

Again we follow our trusty method.

1) Sketch the region
   See Problem 8 (the previous problem)

2) Choose a coordinate system
   As in Problem 8, we pick cylindrical

3) Parametrize the surface of integration
   The difference with problem 8 is that this time, we're integrating <u>on the cylinder</u>, not on the cone.

In cylindrical, we have $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

So on the cylinder, we get $r^2 = 2ar\sin\theta \Rightarrow r = 2a\sin\theta$
($r=0$ is just the z axis, which is included in $r=2a\sin\theta$ anyway)

The cone gives us $z^2=r^2 \Rightarrow z=\pm r$, and so $-r\le z\le r$

Finally, projecting the surface onto the xy plane as in Problem 8 gives $0\le\theta\le\pi$

$$\therefore \vec{r}_s(\theta,z) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}$$
$$= (2a\sin\theta)\cos\theta\,\hat{i} + (2a\sin\theta)\sin\theta\,\hat{j} + z\hat{k}$$
$$= 2a\sin\theta\cos\theta\,\hat{i} + (2a\sin^2\theta)\hat{j} + z\hat{k}$$

where $0\le\theta\le\pi$ and $-r\le z\le r \Rightarrow -2a\sin\theta\le z\le 2a\sin\theta$

Notice that because r is fixed, we have to substitute for it in terms of the other two parameters that are actually varying.

4) Integrand = 1 (we're finding area)

5) Differential element dS: $dS = \left|\frac{\partial \vec{r}_s}{\partial \theta} \times \frac{\partial \vec{r}_s}{\partial z}\right| d\theta\, dz$

$$\frac{\partial \vec{r}_s}{\partial \theta} = (2a[\cos^2\theta - \sin^2\theta])\hat{\imath} + (4a\sin\theta\cos\theta)\hat{\jmath} + 0\hat{k}$$

$$\frac{\partial \vec{r}_s}{\partial z} = 0\hat{\imath} + 0\hat{\jmath} + (1)\hat{k} = \hat{k}$$

Thus $\frac{\partial \vec{r}_s}{\partial \theta} \times \frac{\partial \vec{r}_s}{\partial z} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2a(\cos^2\theta - \sin^2\theta) & (4a\sin\theta\cos\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} \\ 2a(\cos^2\theta - \sin^2\theta) & (4a\sin\theta\cos\theta) \end{bmatrix}$$

$$= [4a\sin\theta\cos\theta]\hat{\imath} - [(2a)(\cos^2\theta - \sin^2\theta)]\hat{\jmath}$$

Hence $\left|\frac{\partial \vec{r}_s}{\partial \theta} \times \frac{\partial \vec{r}_s}{\partial z}\right| = \sqrt{16a^2\sin^2\theta\cos^2\theta + 4a^2(\cos^2\theta - \sin^2\theta)^2}$

$$= 2a\sqrt{4\sin^2\theta\cos^2\theta + \cos^4\theta - 2\sin^2\theta\cos^2\theta + \sin^4\theta}$$

$$= 2a\sqrt{\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta}$$

$$= 2a\sqrt{(\cos^2\theta + \sin^2\theta)^2} = 2a\sqrt{(1)^2}$$

$$= 2a$$

$\therefore\ dS = 2a\, d\theta\, dz$

6) Evaluate the integral:

$$A = \iint_S dS = \int_0^{\pi} d\theta \int_{-2a\sin\theta}^{2a\sin\theta} dz\,(2a)$$

$$= 2a\int_0^{\pi}[4a\sin\theta]\,d\theta = 8a^2\int_0^{\pi}\sin\theta\, d\theta$$

$$= 8a^2[-\cos\theta]_0^{\pi} = 8a^2[-(-1)-(-1)] = 16a^2$$

$\therefore$ the area is $\boxed{16a^2 \text{ units}^2}$

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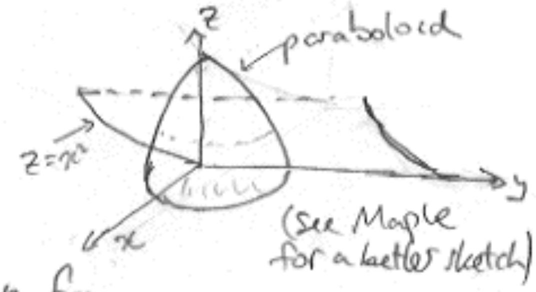
Problem 15 (Section 15.5, Adams)

Question: Find $\iint_S xz\, dS$ where S is the part of the surface $z=x^2$ that lies in the first octant $(x,y,z \geq 0)$ and also inside the paraboloid $z = 1-3x^2-y^2$

Solution:

1) Sketch the region



2) We'll use Cartesian coordinates

3) $\vec{r}_S(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

We have $z=x^2$ on the surface

To find bounds for x and y, observe from the sketch that the intersection of $z=x^2$ and $z=1-3x^2-y^2$ bounds the surface: $x^2 = 1-3x^2-y^2 \Rightarrow 4x^2+y^2=1$

If we let $y=0$, we see that $4x^2=1 \Rightarrow x=1/2$ (since $x\geq 0$)

Also $y=\sqrt{1-4x^2}$ (again since $y\geq 0$) on the intersection

Thus from the sketch, $0\leq x \leq 1/2$ and $0\leq y \leq \sqrt{1-4x^2}$

$\therefore \vec{r}_S(x,y) = x\hat{\imath} + y\hat{\jmath} + x^2\hat{k}$, with $0 \leq x \leq 1/2$ and $0\leq y\leq \sqrt{1-4x^2}$

4) Our integrand is $xz = x(x^2) = x^3$

5) Compute dS: $\partial\vec{r}_S/\partial x = \hat{\imath} + (2x)\hat{k}$; $\partial\vec{r}_S/\partial y = \hat{\jmath}$

$\Rightarrow \partial\vec{r}_S/\partial x \times \partial\vec{r}_S/\partial y = \hat{\imath}\times\hat{\jmath} + (2x)(\hat{k}\times\hat{\jmath}) = \hat{k} - (2x)\hat{\imath}$

$\therefore dS = |\partial\vec{r}_S/\partial x \times \partial\vec{r}_S/\partial y|\,dx\,dy = \sqrt{(-2x)^2+1^2}\,dx\,dy = \sqrt{1+4x^2}\,dx\,dy$

6) Evaluate the integral

$$\begin{aligned}\iint_S xz\,dS &= \int_0^{1/2} dx \int_0^{\sqrt{1-4x^2}} dy\,(x^3)(\sqrt{1+4x^2}) \\ &= \int_0^{1/2} x^3\sqrt{1+4x^2}\left[\sqrt{1-4x^2}-0\right]dx \\ &= \int_0^{1/2} x^3\sqrt{(1+4x^2)(1-4x^2)}\,dx = \int_0^{1/2} x^3\sqrt{1-16x^4}\,dx \\ &= \int_1^0 \sqrt{u}\left(-du/64\right) = (1/64)\left[\frac{u^{3/2}}{3/2}\right]_0^1 \\ &= (1/64)(2/3)\left[1^{3/2}-0^{3/2}\right] \\ &= 1/96\end{aligned}$$

Let $u = 1-16x^4$
Then $du = -16\cdot 4x^3\,dx$
$\Rightarrow x^3\,dx = -du/64$
$x=0 \Rightarrow u=1$
$x=1/2 \Rightarrow u=0$

$$\therefore \iint_S xz\,dS = \boxed{1/96}$$