TUTORIAL #7
The Fundamental Theorems of Calculus

Presented by Sacha Nandlall
T.A. for MATH 264 (Advanced Calculus)

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at 11:35 A.M. in Wilson 103

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IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

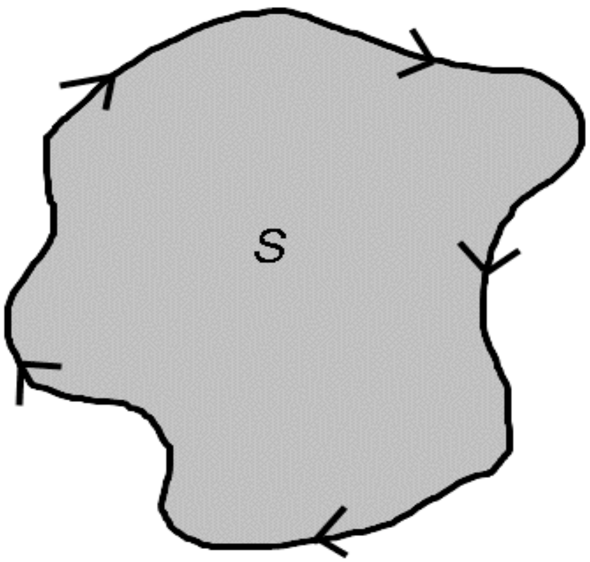
ORIENTATION OF SURFACE BOUNDARIES

- Consider an open (i.e. non-closed) surface S with normal vector $\mathbf{N}$ (and $d\mathbf{S} = \mathbf{N}\, dS$)
- We will denote the curve that forms the boundary of this surface as ∂D
  - This boundary curve can be oriented in two directions, e.g. clockwise (CW) or counterclockwise (CCW)
  - We will distinguish two opposite orientations of ∂D:
    - The "positively-oriented" boundary, denoted ∂D^+
    - The "negatively-oriented" boundary, denoted ∂D^-
- Which orientation is which? Use this rule...

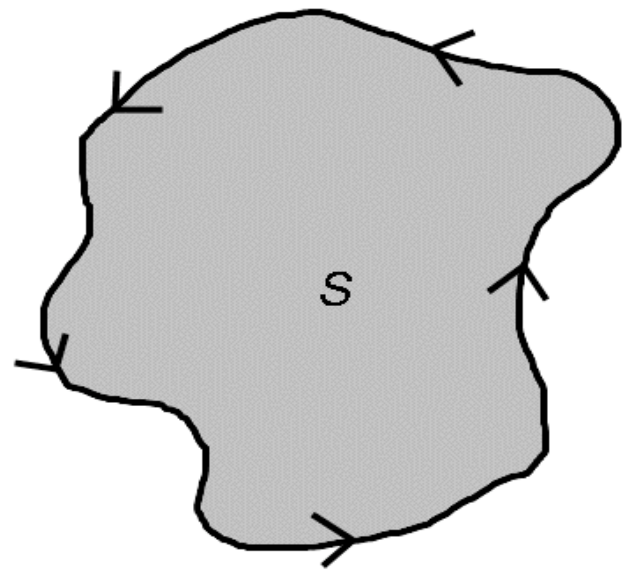
RIGHT-HAND RULE FOR SURFACE BOUNDARIES

1. Point the right hand's thumb in the direction of ***dS*** (or the normal vector ***N***)
2. Place the palm of the right hand against the boundary's surface
3. Then the other fingers of the hand point in the direction corresponding to ∂D^+ (the positively-oriented boundary)

ORIENTING SURFACE BOUNDARIES - DIAGRAMS

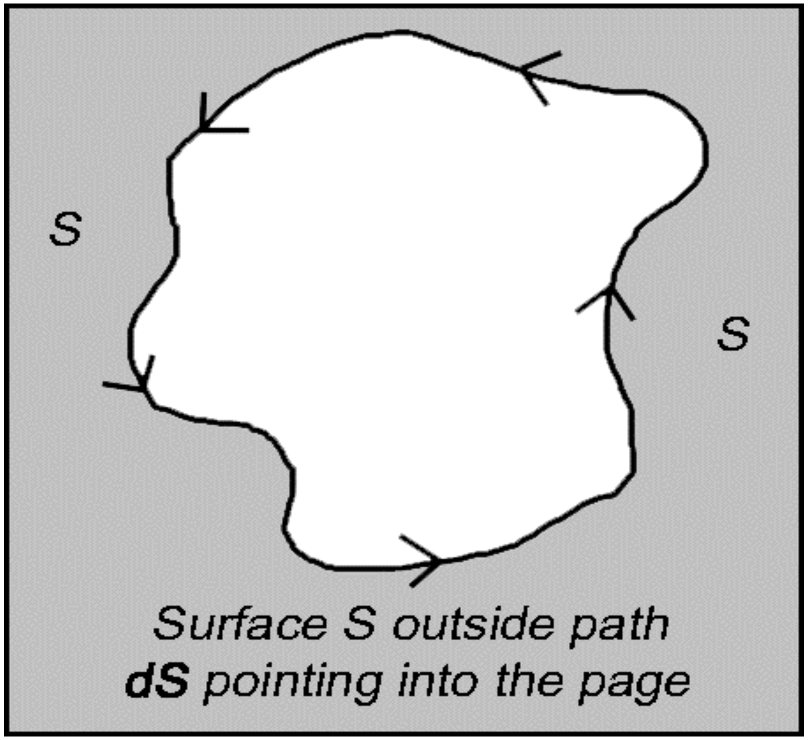


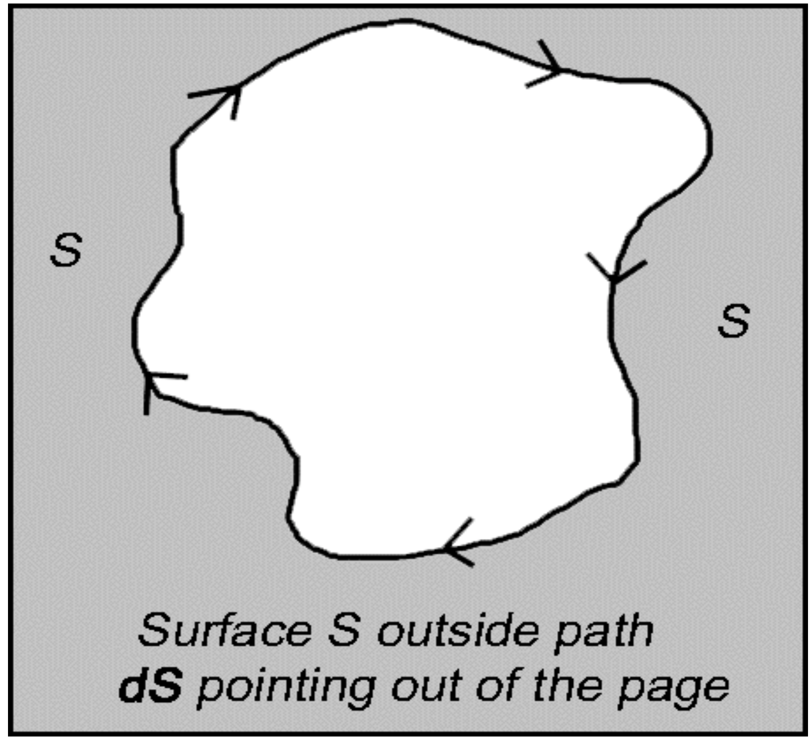
Surface S inside path
***dS** pointing into the page*



Surface S inside path
***dS** pointing out of the page*

ORIENTING SURFACE BOUNDARIES - DIAGRAMS





VERSIONS OF THE FUNDAMENTAL THEOREM

- We'll now list some theorems, which are all in some sense different versions of the fundamental theorem of calculus
- These theorems allow one to simplify (or at least express in a different form) integrals in 1, 2, or 3 dimensions
- Knowing when to apply which theorem is a bit of an art, and takes practice

1D FUNDAMENTAL THEOREMS

- Normal 1D fundamental theorem (from your first integral calculus course!):

$$\int_a^b \left(\frac{df(t)}{dt}\right) dt = f(b) - f(a)$$

- Fundamental theorem for conservative fields
  - Let *C* be a vector curve parametrized as follows: $C \sim \vec{r}(t),\ t \text{ from } a \text{ to } b$
  - Then: $\int_C (\text{grad } \phi) \bullet d\vec{r} = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$

2D FUNDAMENTAL THEOREMS

- Let:
  - *S* be a 2D or 3D surface containing no "holes"
  - ***dS*** = ***N*** *dS* be the vector differential surface element associated with the surface *S*
  - ∂D^+ denote the boundary of surface *S*, oriented positively with respect to the ***dS*** vector
- Stokes' theorem: $\iint_S (\text{curl } \vec{F}) \bullet d\vec{S} = \oint_{\partial D^+} \vec{F} \bullet d\vec{r}$
- Divergence theorem (2D): $\iint_S (\text{div } \vec{F})\, dS = \oint_{\partial D^+} (\vec{F} \bullet \widehat{N}) ds$

2D FUNDAMENTAL THEOREMS

- Green's theorem
  - Special case of Stokes' theorem for functions lying entirely in the *xy* plane (i.e. $F_3 = 0$)
  - Instead of Green's theorem, use Stokes' with:

$$\text{curl } \vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \widehat{k} \qquad d\vec{S} = \pm \widehat{k}\, dS \text{ (i.e. } \widehat{N} = \pm \widehat{k})$$

  - This is recommended for a couple of reasons:
    - You won't get the boundary orientation wrong
    - You can easily compute the curl of a 2D vector function if you forget it (no extra memorization)

3D FUNDAMENTAL THEOREMS

- Let:
  - *V* be any volume in 3D space
  - *S* be the closed surface bounding *V*
  - ***dS*** be the outward-pointing differential surface element associated with the surface *S*
- Divergence theorem (3D):
  - First form (most useful): $\iiint_V (\text{div } \vec{F})\, dV = \oiint_S \vec{F} \bullet d\vec{S}$
  - Second form: $\iiint_V (\text{curl } \vec{F})\, dV = -\oiint_S \vec{F} \times d\vec{S}$
  - Third form: $\iiint_V (\text{grad } \phi)\, dV = \oiint_S \phi\, d\vec{S}$