TUTORIAL #6
More Vector Fields

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T.A. for MATH 264 (Advanced Calculus)

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IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

FIELD LINES

- Consider a 3D field $\vec{F}(x,y,z) = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$
- The field lines of ***F*** are given by: $\dfrac{dx}{F_1} = \dfrac{dy}{F_2} = \dfrac{dz}{F_3}$
  - Also known as flow lines or streamlines
  - There are two differential equations, usually solvable as separable or exact ODEs
    - Note: when using the exact ODEs method for a 3D field, the unknown function that appears is in two variables, not one

VECTOR POTENTIALS

- ***A*** is a vector potential of ***F*** if

$$\vec{F} = \text{curl } \vec{A} = \nabla \times \vec{A}$$

- Use the method of exact ODEs to find ***A***
  - Can make up to 6 "degrees of freedom" worth of assumptions on the unknowns
    - This is because the equation defining ***A*** restricts only 3 of the 9 partial derivatives involved
  - ***A*** isn't unique (many different ***A*** are possible)
    - Check the ***A*** you find using the equation above

SCALAR POTENTIALS AND CONSERVATIVE FIELDS

- $\varphi(x, y, z)$ is a (scalar) potential of ***F*** if

$$\vec{F} = \text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}$$

  – Fields with potentials are said to be conservative
  – To prove that a field is conservative, find a potential for it using the exact ODEs method
- Equipotential surfaces (or curves in 2D)
  – The family of surfaces/curves given by $\varphi(x, y, z) = C$ (C is a constant that can take on any real value)
  – Field lines always intersect equipotentials at 90°

CRITERION FOR CONSERVATIVE FIELDS

- It can be shown that any conservative field must satisfy the criterion: $\text{curl } \vec{F} = \nabla \times \vec{F} = \vec{0}$
  – In 2D, this reduces to $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$
- Caution: just because a field ***F*** satisfies this criterion doesn't mean it's conservative!
  – However, any field that doesn't satisfy it automatically can't be conservative ("necessary but not sufficient")
  – If ***F*** contains arbitrary constants, use this criterion to determine the values of these constants for which ***F*** could be conservative

SOLENOIDAL AND IRROTATIONAL FIELDS

- Two important vector identities:

  $\text{div curl } \vec{F} = 0 \qquad \text{curl grad } \phi = \vec{0}$

- Some terminology:
  - ***F*** is solenoidal if $\text{div } \vec{F} = 0$
  - ***F*** is irrotational if $\text{curl } \vec{F} = \vec{0}$
- So, from the above identities…
  - All fields with vector potentials are solendoial
  - All conservative fields are irrotational