TUTORIAL #5
Line Integrals

Presented by Sacha Nandlall
T.A. for MATH 264 (Advanced Calculus)

Thursday, October 12, 2006
at 11:35 A.M. in Wilson 103

Website: http://www.resanova.com/teaching/calculus/
Email: sacha.nandlall@mail.mcgill.ca

McGill

IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

LINE INTEGRALS

- A line integral is the integral of a scalar or vector function over a 2D or 3D curve
  - Line integrals versus standard single-variable integrals is analogous to integration over 3D surfaces versus integration in the *xy*-plane
- Line integrals of vector functions ***F***:
  - Add up components of ***F*** tangential to the curve
  - If ***F*** is a force, the line integral gives the work done by ***F*** along the curve of integration

FORMS OF LINE INTEGRAL

- Standard scalar form: $\int_C f\,ds$
- Standard vector form: $\int_C \vec{F} \bullet d\vec{r}$
- Expanded notation: $\int_C [f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz]$
  - Integrals in this form should be converted to one of the other two forms as follows:
    - To standard scalar form: break the integral into the sum of three smaller ones with *ds* = *dx*, *dy*, and *dz*
    - To standard vector form: rewrite as a dot product using

$$\vec{F} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k} \qquad d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

TERMINOLOGY AND DIFFERENTIAL ELEMENTS

- *C* is the curve of integration
- *f* (scalar) and ***F*** (vector) are the integrands
- *ds* (scalar) and ***dr*** (vector) are differential elements of curve *C*
  - The vector element ***dr*** is tangential to *C* (***T*** is a unit tangential vector to the curve): $d\vec{r} = \widehat{T}ds$
  - As before, the scalar element *ds* is the magnitude of the vector element: $ds = |d\vec{r}|$

SCALAR VERSUS VECTOR LINE INTEGRALS

- There are two types of line integral
  - Scalar: when the direction of integration along the curve *C* is not specified
  - Vector: when the direction is specified
- Caution: the form the integral is expressed in doesn't tell you what type it is
  - In fact, an integral in any of the three forms can be converted into any one of the other forms

METHOD TO EVALUATE LINE INTEGRALS

1. Rewrite the integral into either standard scalar form or standard vector form
2. Parametrize the curve *C* with a vector function of a single parameter *t*:
   - If the line integral is vector (i.e. direction of integration is specified): $C \sim \vec{r}(t),\, t \text{ from } a \text{ to } b$
     - Parameter *t* can go from smallest to largest values (a < b), or largest to smallest (a > b)
   - If the line integral is scalar: $C \sim \vec{r}(t),\, a \leq t \leq b$
     - Parameter *t* must go from smallest to largest value

METHOD TO EVALUATE LINE INTEGRALS

3. Express the integrand as a function of *t*
4. Write the differential element expression

$$d\vec{r} = \left(\frac{d\vec{r}(t)}{dt}\right) dt \qquad ds = |d\vec{r}| = \left|\frac{d\vec{r}(t)}{dt}\right| dt$$

5. Evaluate the integral (plug in steps 2-4)

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b [\vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}(t)}{dt}]\, dt \qquad \int_C f\, ds = \int_a^b f(t) \cdot \left|\frac{d\vec{r}(t)}{dt}\right| dt$$

PARAMETRIZING 2D AND 3D CURVES

- For a line: take any of the Cartesian coordinate variables *x*, *y*, *z* that changes as a parameter
- "$ax^2 + by^2$" $(a, b > 0)$ suggests $x = \sqrt{b}\cos(t)$, $y = \sqrt{a}\sin(t)$
- For intersections of surfaces:
  - Solve for one of the variables in both surface equations to get one equation in two variables
  - Then, parametrize the two variables left in that single equation in terms of one parameter
  - If one surface's equation depends on only two variables, parametrize that equation first

INTEGRATING OVER UNIONS OF CURVES

- Suppose curve *C* is comprised of the union of *n* simpler curves labelled C_1 to C_n, i.e. $C = C_1 \cup C_2 \cup \cdots \cup C_n = \bigcup_{k=1}^{n} C_k$
- Then line integrals along *C* can be decomposed into a piecewise sum of the individual integrals, e.g.:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds = \sum_{k=1}^{n} \int_{C_k} f\,ds$$

INTEGRATING AROUND CLOSED CURVES

- A curve is closed if it begins where it ends
- The line integral of *f* along a closed curve *C* is called the circulation of *f* along *C*
  - Use a circle to denote circulation, e.g. $\oint_C f\,ds$
  - If the direction of circulation is specified, the line integral is a vector one (if not it's scalar)
    - Changing direction just changes the answer's sign
    - To distinguish counterclockwise (CCW) and clockwise (CW), inscribe an arrow on the circle