TUTORIAL #12
Boundary Value Problems

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T.A. for MATH 264 (Advanced Calculus)
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IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

EVEN AND ODD FOURIER SERIES

- Suppose we want a Fourier series of some function *f(x)* with only sines or cosines
  - The interval is only allowed to include zero as an endpoint (but not as an interior point)
  - Inclusion or exclusion of the endpoints *a* and *b* doesn't matter
  - It's also assumed we don't care what values the Fourier series has outside some interval *[a, b]*

EVEN AND ODD FOURIER SERIES

- Define the odd and even extensions of *f(x)*:

$$f_o(x) = \begin{cases} -f(-x) & \text{if } x \in [-b, -a] \\ f(x) & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases} \qquad f_e(x) = \begin{cases} f(-x) & \text{if } x \in [-b, -a] \\ f(x) & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

- The Fourier series of $f_o(x)$ will contain only sines only, and $f_e(x)$ only cosines
- Moreover, both series will evaluate to *f(x)* in interval *[a, b]* as we wanted
  - Why? Because $f_o(x) = f_e(x) = f(x)$ in *[a, b]*

EVEN AND ODD FOURIER SERIES

- Note that either series equals *f(x)* only in interval *[a, b]* (not necessarily true outside)
- The sine-only series is called either:
  - The Fourier sine series of *f(x)*
  - The quarter-range sine (QRS) expansion of *f(x)*
- The cosine-only series is called either:
  - The Fourier cosine series of *f(x)*
  - The half-range cosine (HRC) expansion of *f(x)*

EVEN AND ODD FOURIER SERIES

- In general, series contain sines only for odd functions, and cosines only for even ones
- If *f(x)* is real, we can avoid doing a complex integral for the series coefficients since:

$$a_n = \mathrm{Re}[c_n] = \mathrm{Re}[\frac{1}{T}\int_T f(x)e^{-in\omega_0 x}\,dx] = \frac{1}{T}\int_T f(x)\cos{(n\omega_0 x)}\,dx$$

$$b_n = \mathrm{Im}[c_n] = \mathrm{Im}[\frac{1}{T}\int_T f(x)e^{-in\omega_0 x}\,dx] = -\frac{1}{T}\int_T f(x)\sin{(n\omega_0 x)}\,dx$$

  - Odd functions have no cosine terms, so $a_n = 0$
  - Even ones have no sine terms, so $b_n = 0$

BOUNDARY VALUE PROBLEMS

- A boundary value problem (BVP) consists of the following:
  - A partial differential equation (PDE) in a function $u(x, t)$ that depends on space and time
  - Initial conditions (ICs) and/or boundary conditions (BCs)
    - BCs give $u(x, t)$ at a fixed point in space $x = x_0$
    - ICs give $u(x, t)$ at a fixed "starting" time $t = t_0$

METHOD TO SOLVE BVPs

1. Clearly define the ingredients of the BVP
   a) Define the unknown function *u(x, t)* and state what physical quantity it represents (e.g. temperature, wave amplitude)
   b) Specify the ranges on *x* and *t*
   c) State the PDE governing the BVP
   d) List the boundary conditions
   e) List the initial conditions

METHOD TO SOLVE BVPs

2. Write down the BVP's general solution
   – Can derive it sometimes by assuming *u(x, t)* is separable, i.e. $u(x,t) = F(x)G(t)$
   – For some BVPs, just memorize the solution
3. Apply the boundary conditions and initial conditions to find all of the free parameters in the general solution

HEAT DIFFUSION BVPs

- Involves finding temperature distribution in a closed region (e.g. a metal rod)
- *u(x, t)* is temperature at point *x*, time *t*
- Governing PDE (the heat equation):

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

- General solution:

$$u(x,t) = C_1 + C_2 x + \sum_{n=1}^{+\infty} [2a_n \cos(n\omega_0 x) - 2b_n \sin(n\omega_0 x)] e^{-n^2\omega_0^2\alpha^2 t}$$

HEAT DIFFUSION BVPs REMARKS

- C_1, C_2, a_n, b_n, and ω_0 are real parameters obtained using the BCs and ICs
- ω_0 must be positive and nonzero
- Linearity: any (nonempty) range of *n* in the summation is also a valid solution
- The steady-state solution to the BVP (i.e. as time tends to infinity) is

$$u_s(x) = \lim_{t \to +\infty} u(x,t) = C_1 + C_2 x$$

WAVE BVPs

- Describes a wave propagating in a medium, e.g. an EM field or mechnical displacement
- *u(x, t)* is wave amplitude at point *x*, time *t*
- Governing PDE (the wave equation):

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

- General solution (d'Alembert formula):

$$u(x,t) = f_+(x-ct) + f_-(x+ct)$$

WAVE BVPs REMARKS

- c is the wave's speed, which must be positive and nonzero
- f_+ and f_- are any two functions
  - $f_+(x - ct)$ is a forward-propagating wave: for a fixed wave amplitude, spatial location x increases as time t increases
  - $f_-(x + ct)$ is a backward-propagating wave: for a fixed wave amplitude, x decreases for increasing t