TUTORIAL #10
Fourier Series

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T.A. for MATH 264 (Advanced Calculus)
Thursday, November 9, 2006
at 11:35 A.M. in Wilson 103

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IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

COMPLEX NUMBERS: BASICS AND FORMS

- Define the imaginary unit *i* so that $i^2 = -1$
- Two ways to write a complex number *z*:
  - Rectangular/Cartesian form: $z = x + iy$
    - $x = Re(z)$ is called the real part of *z*
    - $y = Im(z)$ is called the imaginary part of *z*
  - Polar form: $z = r\angle\theta = re^{i\theta}$
    - $r = |z| = abs(z)$ is called the magnitude of *z*
    - $\theta = arg(z)$ is called the argument/angle/phase of *z*
  - Note that *x*, *y*, *r*, and θ are all real numbers

COMPLEX NUMBERS: FORM CONVERSION

- To convert between rectangular and polar: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{cases}$
  - These are the same formulas used to go from Cartesian to plane polar!
  - Can think of a complex number as a point in the "complex plane" (axes *Re(z)* and *Im(z)*)
- These equations work because the complex exponential is defined by Euler's identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

COMPLEX NUMBERS: FORM CONVERSION

- From Euler's identity, we can come up with alternate definitions of sine and cosine:

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

- Also define the conjugate of a complex number as follows:

$$\overline{z} = z^* = x - iy = r\angle(-\theta) = re^{-i\theta}$$

COMPLEX NUMBERS: OPERATIONS

- Addition/subtraction: Convert to rectangular and add/subtract real and imaginary parts:
$$(a+ib)+(c+id)=(a+c)+i(b+d)$$
- Multiplication: Use either form:
$$(a+ib)(c+id)=(ac-bd)+i(ad+bc)$$
$$(r_1e^{i\theta_1})(r_2e^{i\theta_2})=r_1r_2e^{i(\theta_1+\theta_2)}$$
- Division: Multiply top and bottom by conjugate of bottom: $$\frac{z_1}{z_2}=\frac{z_1}{z_2}\cdot\frac{z_2^*}{z_2^*}=\frac{z_1z_2^*}{|z_2|^2}$$

PERIODIC FUNCTIONS

- *T* is a period of *f(x)* if *f(x)* = *f(x+T)* for all *x*
  - If *T* is a period, so is any multiple of *T*
  - The smallest possible value of *T* is called the fundamental period of *f(x)*
  - The fundamental periods of *sin(x)*, *cos(x)*, and e^{ix} are 2π
- The frequency *f* and angular frequency ω associated with a period *T* are:

$$f = 1/T \qquad \omega = 2\pi f = 2\pi/T$$

FOURIER SERIES

- The Fourier series transformation of a function *f(x)* with fundamental period *T* is:
  - Analysis equation:
    $$c_n = \frac{1}{T}\int_T f(x)e^{-in\omega_0 x}\,dx$$
    - Here the integral means "over any period of *f(x)*" (i.e. over any interval of length *T*), and $\omega_0 = 2\pi/T$
  - Synthesis equation:
    $$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 x}$$
    - The infinite sum is called the Fourier series of *f(x)*

THE FOURIER SERIES COEFFICIENTS

- The c_n are called the Fourier series coefficients or spectral components of *f(x)*
  - The c_n are complex numbers: $c_n = a_n + ib_n$
- c_0 is the average value of the function *f(x)*
  - If *f(x)* is an electrical signal (voltage or current), this is also called the DC offset of bias
- c_1 and c_{-1} form the fundamental of *f(x)*
- c_n and c_{-n} form the n^{th} harmonic of *f(x)*

FOURIER SERIES OF REAL FUNCTIONS

- The Fourier series works for complex *f(x)*
- However, if *f(x)* is real, it can be shown that the c_n are conjugate-symmetric, i.e.:

$$c_n = (c_{-n})^* \Leftrightarrow (a_n = a_{-n}) \wedge (b_n = -b_{-n})$$

- Using this fact and the complex-exponential definitions of *sin* and *cos*, one can rewrite the series of a real function as:

$$f(x) = a_0 + \sum_{n=1}^{+\infty} [2a_n \cos(n\omega_0 x) - 2b_n \sin(n\omega_0 x)]$$

INTERPRETATION OF THE FOURIER SERIES

- The Fourier series expresses $f(x)$ as an infinite sum of sinusoids (sines and cosines)
  - The sinusoid frequencies are multiples of ω_0, i.e. the fundamental frequency of $f(x)$
- Coefficient c_n tells us how strong the frequency component $n\omega_0$ is
- Fourier analysis has tons of applications in all fields of engineering, but all you need to know for this class is how to compute Fourier series