

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 265

ADVANCED CALCULUS

Examiner: Professor A.R. Humphries

Date: December 8, 2003

Associate Examiner: Professor W. Jonsson Time: 9:00 A.M. - 12:00 P.M. hours

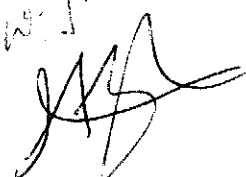
INSTRUCTIONS

**Calculators are not permitted.**

**Answer in exam booklets.**

**This is a closed book exam.**

**Dictionaries are permitted.**

*W. Jonsson*  


This exam comprises the cover and 2 pages of 7 questions.

1. The following equations define  $y$  and  $v$  implicitly as functions of  $x$  and  $u$  near  $(x, y, u, v) = (1, -1, -1, 1)$

$$\begin{aligned} xy + yu^2 + u^2v + x^2v &= 0 \\ x + y^2 + u + v^2 &= 2 \end{aligned}$$

At this choice of values of the variables, compute  $\frac{\partial y}{\partial x}$  and  $\frac{\partial v}{\partial u}$ .

2. With the aid of differentiation under the integral sign show that the if  $x(t)$  satisfies the following integral equation,

$$x(t) + \int_0^t \sin(t - \tau)x(\tau)d\tau = \cos 2t$$

Then it also satisfies the ordinary differential equation

$$x'' + x = -3 \cos 2t \text{ with initial conditions } x(0) = 1 \quad x'(0) = 0$$

3. With the aid of Lagrange Multipliers, find the shortest distance from the origin to a point on the curve of intersection of the surfaces

$$xy + z^2 = 0 \text{ and } x^2 + y^2 = 1$$

4. Let  $C$  be the circle in the  $xy$ -plane of radius  $a = 2$  centered at  $(x, y) = (-1, 3)$

(a) Compute the line integral

$$\oint_C \frac{-(y-3)dx}{(x+1)^2 + (y-3)^2} + \frac{(x+1)dy}{(x+1)^2 + (y-3)^2}$$

where  $C$  is traversed in the positive direction.

- (b) Evaluate the following line integrals (Green's Theorem may be useful).

i.

$$\oint_{\Gamma} \frac{-(y-3)dx}{(x+1)^2 + (y-3)^2} + \frac{(x+1)dy}{(x+1)^2 + (y-3)^2}$$

Where  $\Gamma$  is the boundary of the square  $|x| \leq 2, |y| \leq 4$

ii.

$$\oint_{\Gamma} \frac{-(y-3)dx}{(x+1)^2 + (y-3)^2} + \frac{(x+1)dy}{(x+1)^2 + (y-3)^2}$$

Where  $\Gamma$  is the circumference of the circle of radius  $a = 1$  centered at the origin.

5. Consider the region  $\mathcal{D}$  in the  $u - v$  plane bounded by the circle  $u^2 + v^2 = 1$  where the positive direction around this circle defines the positive direction around the boundary of the surface  $\mathcal{S}$  in  $\mathcal{R}^3$  defined parametrically by

$$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u^2 + v^2)\mathbf{j} + uv\mathbf{k} \text{ for } (u, v) \in \mathcal{D}$$

For the vector field  $\mathbf{F} = (yz + x)\mathbf{i} + (xz + y)\mathbf{j} + (xy + z)\mathbf{k}$ , evaluate  $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$  where  $\partial S$ , the boundary of  $S$ , is traversed in the positive direction.

6. Let  $\rho^2 = x^2 + y^2 + z^2$

(a) Calculate  $\nabla \frac{1}{\rho}$  for  $\rho \neq 0$ .

(b) Show that, if  $\rho \neq 0$ , then  $\nabla \cdot \nabla \frac{1}{\rho} = 0$ .

(c) For the vector field  $\mathbf{F} = -\nabla \frac{1}{\rho}$  and the surface  $S$  of the sphere of radius  $a > 0$ ,  $x^2 + y^2 + z^2 = a^2$ , show that

$$\int \int_S \mathbf{F} \cdot \mathbf{n} dS = 4\pi$$

7. Let  $\mathbf{F} = (y^2 + 2xz, z^2 + 2xy, x^2 + 2yz)$ .

(a) Compute  $\nabla \times \mathbf{F}$ .

(b) Is  $\mathbf{F}$  conservative? If yes, then find a corresponding potential.

(c) Compute  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $C$  is the curve  $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$  for  $0 \leq t \leq 1$ .