DO ALL FOUR (4) QUESTIONS IN PART A, THREE (3) FROM PART B, AND ONE (1) FROM PART C FOR A TOTAL OF EIGHT (8).

PART A: ANSWER EACH OF THE FOLLOWING FOUR (4) QUESTIONS

A.1. Calculate the following line integrals $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{x}$ where

(i) $\mathbf{F}(x,y,z) = (3y, x^2, y+z)$ and $\mathbf{c}$ is the line segment from $(2,1,0)$ to $(4,2,1)$.

(ii) $\mathbf{F}(x,y,z) = (2xy + y^2z^2, x^2 + 2xyz^2 + z, 2xy^2z + y)$ and $\mathbf{c}$ is the curve $\mathbf{x}(t) = (t, t^2, t^3), 0 \le t \le 1$ followed by the line segment from $(1,1,1)$ to $(1,2,3)$.

Which — if any — of the above the line integrals is independent of the path? Explain briefly.

A.2. Compute

$$\int\int\int_V 11y \, dxdydz,$$

where $V = \{(x,y,z) \mid -2 \le x+y-z \le 2, 1 \le 2y+z \le 4, -1 \le 2x-3y+z \le 4\}$.

A.3. Show that if

$$\begin{aligned} 2x + y^2 + u^2v^2 - v &= 3 \\ x^2 + y^2 - v^2 - uv^2 &= 0, \end{aligned}$$

then u and v are functions of x and y near the point $(1,1,1,1)$. Determine

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix},$$

when $x = 1, y = 1, u = 1$, and $v = 1$.

A.4. Consider the solid D that is bounded by $x^2 + y^2 + z^2 = 9, z = 1$ and contains the point $(0,0,2)$.

(i) Calculate the outward flux of $\mathbf{F}(x,y,z) = (x,y,z)$ across the surface S where S is the boundary of D.

(ii) Calculate the outward flux of $\mathbf{F}(x,y,z) = (x,y,z)$ across the piece S_1 of the boundary of D that lies on the sphere $x^2 + y^2 + z^2 = 9$.

(iii) Express the outward flux of $\mathbf{F}(x,y,z) = (x,y,z)$ across the boundary S of D as a triple integral in spherical coordinates. DO NOT EVALUATE IT.

PART B: ANSWER THREE (3) OF THE FOLLOWING QUESTIONS.

B.1. Let $\mathbf{c}_1$ be the boundary of the square with vertices at $(1,1), (3,1), (3,3)$, and $(1,3)$ oriented clockwise. Let $\mathbf{c}_2$ be the boundary of the square with vertices at $(1,1), (-1,1)$, $(-1,-1)$, and $(1,-1)$ also oriented clockwise. Sketch these curves. Calculate the following line integrals, justifying your answers.

(i) $\displaystyle\int_{\mathbf{c}_1} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$

(ii) $\displaystyle\int_{\mathbf{c}_2} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$

B.2. If $\mathbf{c}$ is the boundary of the square with vertices at $(2,0), (0,2), (-2,0)$, and $(0,-2)$, directed counterclockwise, compute

$$\int_{\mathbf{c}} (\sin \pi x + x^2 y)dx + (\cos \pi y + xy^2)dy.$$

B.3. Calculate the line integral $\int_{\mathbf{c}} (2x+3y+x^2)dx + (x-y+z-y^2)dy + (3y-2z-z^2)dz$ where $\mathbf{c}$ is the curve of intersection of the cylinder $y^2+z^2=1$ with the plane $x+y+z=1$, oriented (i.e., directed) counterclockwise when viewed from $(1,0,0)$.

B.4. Let S be the part of the surface of the sphere $x^2+y^2+z^2=4$ for which $x^2+y^2 \geq 1$. Calculate

$$\int\int_S \{(\nabla \times \mathbf{F}) \cdot \mathbf{N}\} dS \text{ — also denoted by } \int\int_S \{(\nabla \times \mathbf{F}) \cdot \mathbf{n}\} dS \text{ —}$$

if $\mathbf{F}(x,y,z) = -yi + x\mathbf{j} + xyz\mathbf{k}$.

B.5. (1) Show that, if $r = \sqrt{x^2+y^2+z^2}$, then $\nabla \cdot \nabla(\frac{1}{r}) = 0$. (Recall that $\nabla \cdot \nabla(\frac{1}{r})$ is also denoted by either $\nabla^2(\frac{1}{r})$ or $\Delta(\frac{1}{r})$.)
(2) Calculate the outward flux of $-\nabla(\frac{1}{r})$ across the boundary of the solid bounded by the ellipsoid $x^2+4y^2+9z^2=36$.

PART C: Answer one (1) of the following questions.

C.1. (1) Determine the dimensions of the largest rectangular box with sides parallel to the coordinate planes that can be placed inside the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$.

(2) How can you deduce your answer from the observation that, when the ellipsoid is replaced by a sphere $x^2 + y^2 + z^2 = 1$, the box with largest volume is a cube?

C.2. Verify that, if f is a scalar function and $\mathbf{F}$ is a vector field,

$$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \nabla f \cdot \mathbf{F}.$$

[Note that $f\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$.]
Use this identity to calculate $\nabla \cdot \mathbf{G}$ when $\mathbf{G}(x, y, z) = \cos(x + y + z)\{yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}\}$.