MATH 264B Advanced Calculus, Winter 2006

Assignment 6 and solution outlines

1. Compute the Fourier series of $f(x) = |\cos x|,\ x \in \mathbb{R}$.

Solution. f is π-periodic, so we set $2l = \pi$. Also, f is even, hence, $b_n = 0$ for all n. Since $|\cos x| = \cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\ dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx = \frac{2}{\pi},$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\ dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cos(2nx)\ dx$$

$$= \left\{\cos x \cos(2nx) = \frac{1}{2}\left(\cos(2n+1)x + \cos(2n-1)x\right)\right\}$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2n+1)x + \cos(2n-1)x)\ dx = \frac{1}{\pi}\left(\frac{\sin(2n+1)x}{2n+1} + \frac{\sin(2n-1)x}{2n-1}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

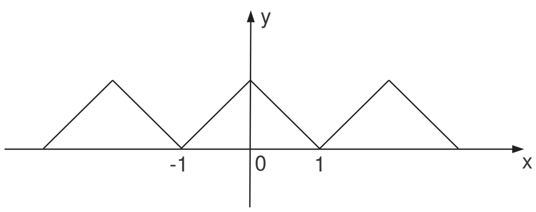
$$= \frac{4}{\pi}\frac{(-1)^{n+1}}{4n^2-1}.$$

Eventually,

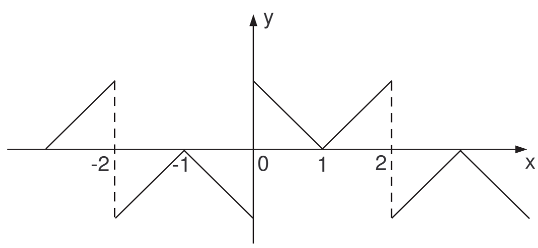
$$\text{FS of} f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(2nx).$$

2. Find HRC and QRS expansions for $f(x) = 1 - x$ on $[0, 1]$.

Solution. $L = 1$



HRC



QRS

a) HRC of f is even, so $b_n = 0$ and

$$a_0 = \int_0^1 (1-x)\ dx = \frac{1}{2},$$

$$a_n = 2\int_0^1 (1-x)\cos(n\pi x)\ dx = 2\left(\int_0^1 \cos(n\pi x)\ dx - \int_0^1 x\cos(n\pi x)\ dx\right)$$
$$= 2\left(\frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 - \left(\frac{1}{n\pi}x\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\ dx\right)\right)$$
$$= \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\ dx = \frac{2}{\pi^2}\cdot\frac{1-(-1)^n}{n^2}.$$

Hence,

$$\text{HRC of} f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos(n\pi x).$$

b) QRS is odd, so $a_n = 0$ for all n and $b_n = 0$ for even n. For odd n we have

$$b_n = 2\int_0^1 (1-x)\cos\frac{n\pi x}{2}\ dx = 2\left(\int_0^1 \sin\frac{n\pi x}{2}\ dx - \int_0^1 x\sin\frac{n\pi x}{2}\ dx\right)$$
$$= 2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1 - \left(-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi x}{2}\ dx\right)\right)$$
$$= \frac{4}{n\pi}\left(1 - \int_0^1 \cos\frac{n\pi x}{2}\ dx\right) = \frac{4}{n\pi}\left(1 - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right).$$

Hence, setting $n = 2k-1$ we have

$$\text{QRS of} f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}.$$

3. Solve the diffusion equation

$$u_{xx} = u_t$$
$$u(0,t) = 0,\ u(10,t) = 100,\ t \in (0,+\infty)$$
$$u(x,0) = 0,\ x \in (0,10)$$

using the method of separation of variables.

Solution. We are looking for a solution in the form

$$u(x,t) = X(x)T(t),$$

which after standard manipulations (see Section 18.3, pp.954-955 in Grinberg) is reduced to

$$u(x,t) = H + Ix + (J\cos\kappa x + K\sin\kappa x)e^{-\kappa^2 t},$$

where H, I, J, K, κ are constants to be determined and $\kappa \neq 0$.

Applying the boundary condition $u(0,t) = 0$ we get $0 = H + Je^{-\kappa^2 t}$ for all $t \in (0,+\infty)$, which is possible only if $H = J = 0$.

Similarly, applying the boundary condition $u(10,t) = 100$ we get $10I = 100$ and $K \sin 10\kappa = 0$. Observe that if $K = 0$ then $u(x,t) = 10x$ and it can't satisfy the initial condition $u(x,0) = 0,\ x \in (0,10)$. Hence, $\sin 10\kappa = 0$ which implies $\kappa = \frac{n\pi}{10},\ n = 0, \pm 1, \pm 2, \ldots$ Superimposing terms $\sin\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t}$ for all integer numbers n (it is enough to consider only natural numbers n) we obtain the formal solution

$$u(x,t) = 10x + \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t},$$

in which we have to determine coefficients K_n.

Applying the initial condition $u(x,0) = 0$ to the formal solution we get

$$-10x = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{10}$$

from which it follows that K_n's are the Fourier coefficients of the HRS expansion of $F(x) = -10x$. Thus,

$$K_n = \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{10}\int_0^{10} -10x\sin\frac{n\pi x}{10}\,dx = \frac{200}{n\pi}\cos n\pi = \frac{200}{n\pi}(-1)^n.$$

Finally,

$$u(x,t) = 10x + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t}.$$

4. Solve the diffusion equation

$$u_{xx} = u_t$$

$$u(0,t) = 25,\ u_x(4,t) = 0,\ t \in (0,+\infty)$$

$$u(x,0) = 25,\ x \in (0,4)$$

using the method of separation of variables.

Solution. As in the previous example we are looking for a solution in the form

$$u(x,t) = H + Ix + (J\cos\kappa x + K\sin\kappa x)e^{-\kappa^2 t},$$

where H, I, J, K, κ are constants to be determined and $\kappa \neq 0$.

Applying the boundary condition $u(0,t) = 25$ we get $25 = H + Je^{-\kappa^2 t}$ for all $t \in (0,+\infty)$, which is possible only if $J = 0,\ H = 25$. Thus,

$$u(x,t) = 25 + Ix + K\sin(\kappa x)e^{-\kappa^2 t}.$$

We have

$$u_x(x,t) = I + K\kappa\cos(\kappa x)e^{-\kappa^2 t},$$

and applying the boundary condition $u_x(4,t) = 0$ we get $0 = I + K\kappa \cos 4\kappa e^{-\kappa^2 t}$ for all $t \in (0, +\infty)$, which implies $I = 0$ and $K\kappa \cos 4\kappa = 0$. If $K = 0$ then $u(x,t) = 25$ is a solution. If $\kappa \cos 4\kappa = 0$ then $\cos 4\kappa = 0$ since $\kappa \neq 0$, so $\kappa = \pm\frac{n\pi}{8}$ for odd natural numbers n (since $\cos(-x) = \cos x$ it is enough to assume $\kappa = \frac{n\pi}{8}$ for odd natural numbers n). Superimposing the solution $u(x,t) = 25$ with the terms $\sin\left(\frac{n\pi x}{8}\right) e^{-\left(\frac{n\pi}{8}\right)^2 t}$ we obtain the formal solution

$$u(x,t) = 25 + \sum_{n=1,3,\ldots}^{\infty} K_n \sin\left(\frac{n\pi x}{8}\right) e^{-\left(\frac{n\pi}{8}\right)^2 t},$$

in which we have to determine coefficients K_n.

Applying the initial condition $u(x,0) = 0$ to the formal solution we get

$$0 = \sum_{n=1,3,\ldots}^{\infty} K_n \sin\frac{n\pi x}{8},$$

which implies $K_n = 0$ for every odd n. Hence,

$$u(x,t) = 25$$

is the general solution.