MATH 264B Advanced Calculus, Winter 2006

Assignment 5, due in class on Thursday March 30, 2006, Solution **outlines**.

1. By Stokes

$$\iint_S \mathbf{G}\, d\mathbf{S} = \iint_S \nabla \times \mathbf{F}\, d\mathbf{S} = \oint_{\partial S^+} \mathbf{F}\, d\mathbf{r}.$$

Now

$$\oint_{\partial S^+} \mathbf{F}\, d\mathbf{r} = \oint_{c_1} \mathbf{F}\, d\mathbf{r} + \oint_{c_2} \mathbf{F}\, d\mathbf{r},$$

where c_1 and c_2 are the circular rims of the cylinder S, parametrized by

$$\mathbf{c}_1(t) = (\cos t, 2, \sin t), \quad \mathbf{c}_2(t) = (\cos t, -2, -\sin t), \quad 0 \le t \le 2\pi.$$

We have

$$\oint_{c_1} \mathbf{F}\, d\mathbf{r} = \int_0^{2\pi} (2, -\cos t, 2\cos t \sin t).(-\sin t, 0, \cos t)\, dt = 0.$$

Likewise,

$$\oint_{c_2} \mathbf{F}\, d\mathbf{r} = \int_0^{2\pi} (-2, -\cos t, 2\cos t \sin t).(-\sin t, 0, -\cos t)\, dt = 0.$$

2. We have

$$\mathbf{F} = (2(y-z), 2(z-x), 2(x-y)),\ \nabla \times \mathbf{F} = (-2, -2, -2),$$

and by Stokes

$$\oint_c \mathbf{F}\, d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\, d\mathbf{S}$$

where S is the disk $x^2 + y^2 \le \frac{1}{4}$, $z = \frac{\sqrt{3}}{2}$, oriented with the upward-pointing normal, given by $(0, 0, 1)$. We have therefore

$$\iint_S \nabla \times \mathbf{F}\, d\mathbf{S} = \iint_S (-2, -2, -2).(0, 0, 1)\, dS = -2Area(S) = -\frac{\pi}{2}.$$

3.a) Since $(0, 0, 0)$ lies inside the surface S_a of the sphere of radius a centered at the origin, we obtain by the Gauss Lemma and the divergence theorem

$$\iint_{S_a} \mathbf{F}\, d\mathbf{S} = -4\pi + \iiint_{B_a} 6(x+y+z)\, dxdydz,$$

where B_a denotes the ball of radius a centered at $(0, 0, 0)$. By symmetry, the triple integral is equal to zero, so that the final answer is -4π.

3.b) Since $(0, 0, 0)$ lies outside the surface C of the cube, we obtain by the Gauss Lemma and the divergence theorem

$$\iint_C \mathbf{F}\, d\mathbf{S} = \iiint_{int(C)} 6(x+y+z)\, dxdydz,$$

where $int(C)$ denotes the interior of the cube. We have

$$\iiint_{int(C)} 6(x+y+z)\, dxdydz = \int_1^2 6x\, dx + \int_1^2 6y\, dy + \int_1^2 6z\, dz = 27.$$