MATH 264B Advanced Calculus, Winter 2006

Assignment 2. Solution outlines.

1. Denote the region by D. Hence,

$$\text{volume of } D = \int_D dV = 2\int_{D_1} dV,$$

where D_1 is the part of D lying in the 1st octant. Using the spherical coordinates we get

$$2\int_{D_1} dV = 2\int_0^{\frac{\pi}{4}} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi \int_1^4 \rho^2 \sin\phi \, d\rho = \frac{21\sqrt{2}\pi}{4}.$$

2. The integral curves of $\mathbf{F}(x, y, z)$ are described by the system

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{xy}.$$

Solving this system we get

$$\begin{cases} x^2 + y^2 = C_1 \\ \frac{y^2}{2} = z + C_2, \end{cases}$$

which for any fixed $C_1 \geq 0,\ C_2 \in \mathbb{R}$ describes the intersection of cylinders $x^2+y^2 = C_1$ and $\frac{y^2}{2} = z + C_2$.

3. Observe that $\mathbf{F}(x, y, z)$ is defined everywhere in $\mathbb{R}^3$, and has continuous partial derivatives by x, y and z of any order. To show that $\mathbf{F}(x, y, z)$ is conservative it is enough to find $\phi(x, y, z)$, which is smooth over $\mathbb{R}^3$, and such that $\nabla\phi = \mathbf{F}$.

We have $\mathbf{F}(x, y, z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, where

$$F_1 = yz(2x + y + z),\ F_2 = xz(x + 2y + z),\ F_3 = xy(x + y + 2z).$$

We are looking for a function $\phi(x, y, z)$ such that

$$\frac{\partial\phi}{\partial x} = F_1,\ \frac{\partial\phi}{\partial y} = F_2,\ \frac{\partial\phi}{\partial z} = F_3. \tag{1}$$

Integrating the first equation of (1) we get

$$\phi(x, y, z) = x^2yz + xy^2z + xyz^2 + C(y, z), \tag{2}$$

where $C(y, z)$ is an unknown function. We substitute the obtained expression for ϕ into the second equation of (1). So,

$$x^2z + 2xyz + xz^2 + \frac{\partial C}{\partial y} = xz(x + 2y + z),$$

and we get $\frac{\partial C}{\partial y} = 0$. It follows that $C(y, z) = C(z)$ is a function of z only. Finally, we substitute (2) into the third equation of (1) and get

$$x^2 y + xy^2 + 2xyz + \frac{dC}{dz} = xy(x + y + 2z).$$

It follows that $\frac{dC}{dz} = 0$ and $C(z) = const$. Thus, we have

$$\phi(x, y, z) = xyz(x + y + z) + C, \ C = const.$$

Observe that $\phi(x, y, z)$ is smooth over $\mathbb{R}^3$, and $\nabla\phi = \mathbf{F}$, so $\phi(x, y, z)$ is a potential for $\mathbf{F}(x, y, z)$.

4. $\mathbf{F}(x, y, z)$ is defined in the region $D = \{(x, y, z) \mid y \neq -z\}$, moreover, $\mathbf{F}(x, y, z)$ is smooth in D. To find the equipotential surfaces we have to find a potential $\phi(x, y, z)$ for $\mathbf{F}(x, y, z)$. Similarly to the previous problem we get

$$\phi(x, y, z) = \frac{2x}{(y + z)^{\frac{1}{2}}} + C, \quad C = const.$$

Observe that $\phi(x, y, z)$ is smooth in D, so it is a potential for $\mathbf{F}(x, y, z)$ in D. Hence, the equipotential surfaces of $\mathbf{F}(x, y, z)$ are described by the equation

$$C_1 x^2 = y + z, \ C_1 \geq 0.$$

It is easy to see that for any fixed $C_1 > 0$ the surface described by this equation is a parabolic cylinder with the straight line $x = 0, \ y = -z$ eliminated.

5. $\mathcal{C}$ is parametrized as

$$\mathbf{r}(t) = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},$$

so we have

$$\frac{d\mathbf{r}}{dt} = a(1 - \cos t)\mathbf{i} + a \sin t\mathbf{j},$$

and

$$\left|\frac{d\mathbf{r}}{dt}\right| = a\sqrt{2(1 - \cos t)}.$$

Hence,

$$\int_{\mathcal{C}} y^2 \, ds = \sqrt{2}a^3 \int_0^{2\pi} (1 - \cos t)^{\frac{5}{2}} \, dt = \left\{1 - \cos t = 2\sin^2 \frac{t}{2}\right\} = 8a^3 \int_0^{2\pi} \sin^5 \frac{t}{2} \, dt$$

$$= 8a^3 \int_0^{2\pi} \sin \frac{t}{2} \left(1 - \cos^2 \frac{t}{2}\right)^2 dt = \left\{u = \cos \frac{t}{2}\right\} = -16a^3 \int_1^{-1} (1-u^2)^2 \, du = \frac{256a^3}{15}.$$