Department of Mathematics and Statistics, McGill University
MATH 265 ADVANCED CALCULUS: ASSIGNMENT 5

This assignment is due in class on Thursday, March 25, 2004.

1. Verify that the vector field $\mathbf{F}(x, y, z) = (x^2+yz)\mathbf{i}+(y^2+zx)\mathbf{j}-2z(x+y)\mathbf{k}$ is solenoidal and determine a corresponding vector potential.

2. Newton's law of motion applied to a perfect fluid is $\rho\mathbf{a} = \nabla p$. This equation relates the density $\rho(x, y, z)$ and pressure $p(x, y, z)$ to the acceleration $\mathbf{a}(x, y, z)$ at a point (x, y, z). Verify the following identities:

   (a) $\mathbf{a} \cdot (\nabla \times \mathbf{a}) = 0$, (b) $\nabla \times (\rho\mathbf{a}) = \mathbf{0}$, (c) $\rho(\nabla \times \mathbf{a}) = \mathbf{a} \times \nabla\rho$.

3. (Adams 16.3.2,4) Evaluate each of the following:

   (a) $\oint_{\mathcal{C}}(x^2 - xy)\, dx + (xy - y^2)\, dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 1)$ and $(2, 0)$;

   (b) $\oint_{\mathcal{C}} x^2y\, dx - xy^2\, dy$, where $\mathcal{C}$ is the clockwise boundary of the region $0 \leq y \leq \sqrt{9 - x^2}$.

4. (Adams 16.4.10) Let $\phi(x, y, z) = xy + z^2$. Find the flux of $\nabla\phi$ upward through the triangular planar surface S with vertices at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, with a,b and c all positive.

5. (Adams 16.4.14) Evaluate $\int\int_S (3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \cdot d\mathbf{S}$, where S is that part of the cylinder $y^2 + z^2 = 1$ which lies in the first octant and between the planes $x = 0$ and $x = 1$.

6. Given the vector field $\mathbf{F} = (x^2 + y^2 + z^2)^{-3/2}(x, y, z) + \nabla \times (\cos x, y^3 \sin xy, z)$ compute $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ where S is the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1.$$

March 15, 2004.