Department of Mathematics and Statistics, McGill University
MATH 265 ADVANCED CALCULUS: ASSIGNMENT 4

This assignment is due in class on Tuesday, March 16, 2004.

1. (Adams 15.6.6) Find the flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward throught the part of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$.

2. (Adams 15.6.16) Find the flux of the plane vector field $\mathbf{F}(x, y) = -\dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$ inward across each of the two curves:

   (a) the circle $x^2 + y^2 = a^2$;

   (b) the boundary of the square $-1 \le x, y \le 1$.

3. Let $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)^n(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\int\int_S \mathbf{F} \cdot \mathbf{N} dS$ where S is the boundary of the spherical shell consisting of points (x, y, z) satisfying $a^2 < x^2 + y^2 + z^2 < b^2$ where $0 < a < b$.

4. Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j}$. Evaluate $\int\int_S \mathbf{F} \cdot \mathbf{N} dS$ where S is the boundary of the region between $z = 0$ and $z = h$ with $x^2 + y^2 < a^2$, where a and h are positive.

5. Let $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Calculate the flux over the sphere of radius a centered at the origin and over the cube $-a < x < a.\ -a < y < a,\ -a < z < a$. You should get the same result and it should be independent of a!

6. Verify the following identities:

   (a) $\nabla \cdot (\phi\mathbf{F}) = (\nabla\phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$;

   (b) $\nabla.(\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$;

   (c) $\nabla \times (\nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$.

7. Consider $\phi(x, y, z) = xyz + y^2$, $\mathbf{F}(x, y, z) = (xy, z^2, y)$ and $\mathbf{G}(x, y, z) = (xy, yz, xz)$. Evaluate each of the following:

   (a) $\nabla \cdot \mathbf{F}$; (b) $\nabla \cdot \mathbf{G}$; (c) $\nabla \times \mathbf{F}$; (d) $\nabla \times \mathbf{G}$;
   (e) $\nabla \cdot (\phi\mathbf{F})$; (f) $\nabla \cdot (\phi\mathbf{G})$; (g) $\nabla \times (\phi\mathbf{F})$; (h) $\nabla \times (\phi\mathbf{G})$;
   (i) $\nabla \cdot (\mathbf{F} \times \mathbf{G})$; (j) $\nabla \times (\mathbf{F} \times \mathbf{G})$; (k) $\nabla(\mathbf{F} \cdot \mathbf{G})$.

8. (Adams 16.2.13/15)

   (a) If ϕ and ψ are smooth scalar fields show that

   $$\nabla \times (\phi\nabla\psi) = -\nabla \times (\psi\nabla\phi) = \nabla\phi \times \nabla\psi$$

   .

   (b) If the vector fields $\mathbf{F}$ and $\mathbf{G}$ are smooth and conservative show that $\mathbf{F} \times \mathbf{G}$ is solenoidal. Find a vector potential for $\mathbf{F} \times \mathbf{G}$.

February 15, 2004.