Solutions to Assignment 2

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Question 1

Question: Find the volume in the upper half space $z \geq 0$ below the cone $x^2 + y^2 = z^2$ inside the cylinder $x^2 + y^2 = 2x$.

Solution:
Denote the region described above as $\mathcal{R}$. We want to find its volume, which is $\iiint_{\mathcal{R}} dV$. We need simply apply the six-step method described in the Tutorial 1 slides to evaluate the integral.

1. *Sketch the region*
   The sketch is similar to the one for Question 2, with the x and y axes interchanged.

2. *Choose a coordinate system*
   For the same reasons given in Question 2, we'll choose cylindrical.

3. *Parametrize the region of integration $\mathcal{R}$*
   This means, find bounds on r, θ, and z. From the sketch, we can see that z goes from the xy-plane ($z = 0$) to the cone. The cone in cylindrical is given by

   $$x^2 + y^2 = z^2 \Rightarrow r^2 = z^2 \Rightarrow z = \pm r \Rightarrow z = r$$

   where the last implication follows since $z \geq 0$. Thus $0 \leq z \leq r$.

To obtain the bounds on r and θ, project the region onto the xy-plane (this is the standard trick for obtaining bounds on r and θ in cylindrical). From the sketch, the projection will simply be the disc bounded by the circle corresponding to the wall of the cylinder, which is drawn in Figure 1). The equation for this circle is the same as that of the cylinder:

$$x^2 + y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$$

Hence $0 \leq r \leq 2\cos\theta$. Moreover, the drawing in Figure 1 shows that $-\pi/2 \leq \theta \leq \pi/2$.

Therefore, the complete parametrization for the region $\mathcal{R}$ is:

$$\mathcal{R} : \begin{cases} 0 \leq r \leq 2\cos\theta \\ -\pi/2 \leq \theta \leq \pi/2 \\ 0 \leq z \leq r \end{cases}$$

4. *Write out the integrand*
   The integrand is simply 1.

5. *Write the expression for differential element*
   The Jacobian determinant for Cartesian-to-cylindrical is r, so:

$$dV = r\,dr\,d\theta\,dz$$

6. *Evaluate the integral*
   Just plug in everything from steps 3 through 5 (continued after next page).

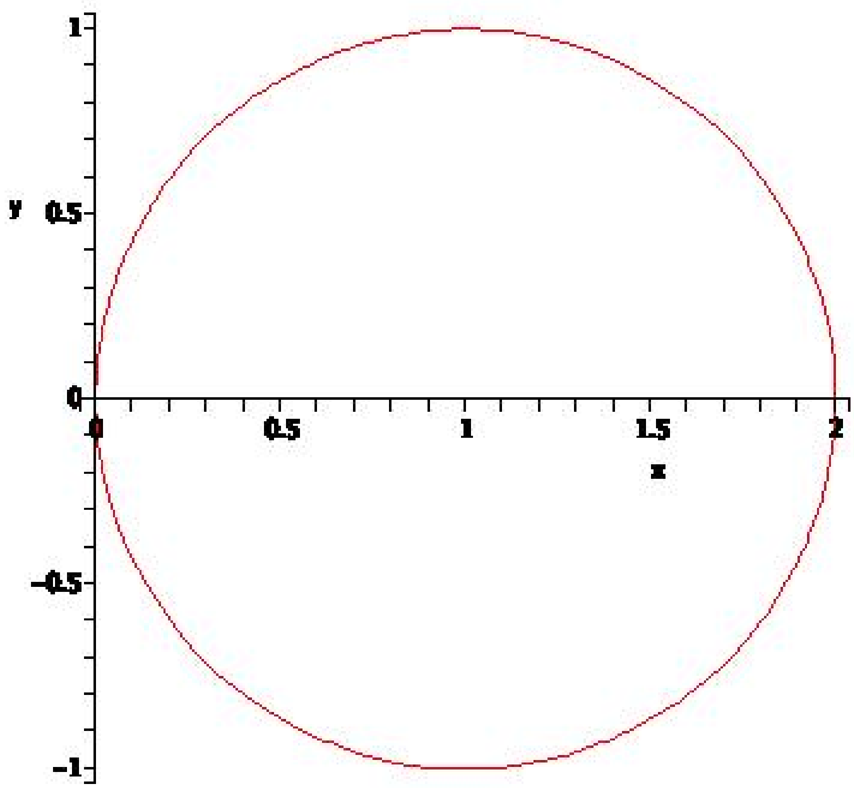

Figure 1: xy-plane projection of the region of integration in Question 1, used to get bounds on the cylindrical coordinates r and θ.

$$
\begin{aligned}
\iiint_R dV &= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} dr \int_0^r dz\, r \\
&= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} dr\, r \int_0^r dz \\
&= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} dr\, r\, [z]_{z=0}^{r} \\
&= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} dr\, r^2 \\
&= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} r^2\, dr \\
&= \int_{-\pi/2}^{\pi/2} d\theta \left[\frac{r^3}{3}\right]_{r=0}^{2\cos\theta} \\
&= \frac{1}{3}\int_{-\pi/2}^{\pi/2} d\theta\, (2\cos\theta)^3 \\
&= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta \\
&= \frac{8}{3}\int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta
\end{aligned}
$$

Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$. Also $\theta = -\pi/2 \Rightarrow u = \sin(-\pi/2) = -1$ and $\theta = \pi/2 \Rightarrow u = \sin(\pi/2) = 1$. Hence:

$$
\begin{aligned}
\iiint_R dV &= \frac{8}{3}\int_{-1}^{1} (1-u^2)\, du \\
&= \frac{8}{3}\left[u - \frac{u^3}{3}\right]_{u=-1}^{1} \\
&= \frac{8}{3}\,[(1-1/3) - (-1+1/3)] \\
&= \frac{8}{3}\cdot\frac{4}{3} \\
&= \frac{32}{9}
\end{aligned}
$$

Therefore the volume of the region is $(32/9)$ *units*^{3}.

Question 2

Question: Find the area of the surface of the cone $x^2 + y^2 = z^2$ which lies in the upper half space $z \geq 0$ and inside the cylinder $x^2 + y^2 = 2y$.

Solution:
This problem is almost identical to Problem 8 of Section 15.5 in the Adams book. A complete solution to this problem is provided in the Tutorial 3 solutions (see the tutorials page at the website listed above); the solutions manual of the Adams book also provides a worked solution (as it does for all even-numbered problems).

The only differences between the problem in Adams and this one are:

- The parameter a in the Adams problem should be set to 1.
- The problem in Adams counts the area of the cone that lies inside the cylinder and below the xy-plane as well, whereas this question asks only for the part of the cone above the xy-plane. Because of the symmetry involved, all this means is the answer to the Adams problem must be divided by 2.

The area of the region is $\sqrt{2}\pi$ *units*^{2}.

Question 3

Question: Using spherical polar co-ordinates, calculate

$$\overline{z} = \frac{\displaystyle\iiint_{\mathcal{R}} z\, dV}{\displaystyle\iiint_{\mathcal{R}} dV}$$

where dV is the element of volume and $\mathcal{R}$ is the region in the upper half space $z \geq 0$ inside the sphere of radius $a > 0$ centered at the origin which lies above the cone $x^2 + y^2 = z^2$.
(Note: There is a mistake in the original assignment: $x^2 + y^2 = z^2$ is a cone, not a cylinder.)

Solution:
To evaluate both integrals, we apply the six-step method as we did in Question 1.

1. *Sketch the region*
   It turns out the region is shaped like an ice-cream cone: it is comprised of a cone with a spherical "cap" on top of it. The sketch is shown in Figure 2.

2. *Choose a coordinate system*
   The question tells us to use spherical.

3. *Parametrize the region of integration $\mathcal{R}$*
   This region is very easy to parametrize in spherical:

$$\mathcal{R} : \begin{cases} 0 \leq \rho \leq a \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/4 \end{cases}$$

4. *Write out the integrands*
   The integrands are $z = \rho \cos \varphi$ and 1.

5. *Write the expression for differential element*
   The Jacobian determinant for Cartesian-to-spherical is $\rho^2 \sin \varphi$, so:

$$dV = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$$

6. *Evaluate the integrals*
   Just plug in everything from steps 3 through 5:

$$\begin{aligned}
\iiint_R dV &= \int_0^a d\rho \int_0^{2\pi} d\theta \int_0^{\pi/4} d\varphi \, \rho^2 \sin \varphi \\
&= \left(\int_0^a \rho^2 d\rho \right) \cdot \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^{\pi/4} \sin \varphi d\varphi \right) \\
&= \left[\frac{\rho^3}{3} \right]_{\rho=0}^{a} \cdot (2\pi - 0) \cdot [-\cos \varphi]_{\varphi=0}^{\pi/4} \\
&= \frac{a^3}{3} \cdot 2\pi \cdot (1 - \frac{\sqrt{2}}{2}) \\
&= \frac{(2 - \sqrt{2})\pi a^3}{3}
\end{aligned}$$

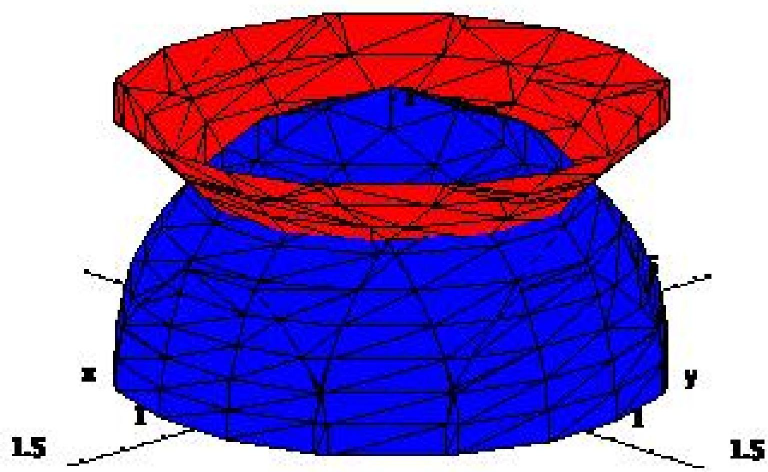

Figure 2: The region of integration for Question 3, with the sphere sketched in blue (shown with radius 1 for illustrative purposes, i.e. $a = 1$) and the cone sketched in red.

$$\begin{aligned}\iiint_R z\,dV &= \int_0^a d\rho \int_0^{2\pi} d\theta \int_0^{\pi/4} d\varphi\,(\rho^2 \sin\varphi)\cdot(\rho\cos\varphi) \\ &= \left(\int_0^a \rho^3 d\rho\right)\cdot\left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^{\pi/4} \sin\varphi\cos\varphi d\varphi\right)\end{aligned}$$

Let $u = \sin\varphi$. Then $du = \cos\varphi\, d\varphi$. Also $\varphi = 0 \Rightarrow u = \sin 0 = 0$ and $\varphi = \pi/2 \Rightarrow u = \sin(\pi/4) = \sqrt{2}/2$. Hence:

$$\begin{aligned}\iiint_R z\,dV &= \left[\frac{\rho^3}{3}\right]_{\rho=0}^{a} \cdot (2\pi - 0)\cdot\left(\int_0^{\sqrt{2}/2} u\,du\right) \\ &= \frac{a^3}{3}\cdot 2\pi \cdot \left[\frac{u^2}{2}\right]_{u=0}^{\sqrt{2}/2} \\ &= \frac{2\pi a^3}{3}\cdot\frac{1}{2}\cdot(\frac{2}{4} - 0) \\ &= \frac{\pi a^4}{8}\end{aligned}$$

So we obtain:

$$\begin{aligned}\overline{z} &= \frac{\pi a^4}{8}\cdot\frac{3}{(2-\sqrt{2})\pi a^3} \\ &= \frac{3a}{8\cdot(2-\sqrt{2})} \\ &= \frac{3a}{8\cdot(2-\sqrt{2})}\cdot\frac{2+\sqrt{2}}{2+\sqrt{2}} \\ &= \frac{3a(2+\sqrt{2})}{16}\end{aligned}$$

Therefore $\overline{z} = 3a(2+\sqrt{2})/16$ *units.*

As an aside, $\overline{z}$ actually represents the z-coordinate of the center of mass for the region $\mathcal{R}$ assuming it has uniform density. This gives additional confirmation for our answer, because we'd expect the center of mass for this shape to lie inside of it (meaning $0 \leq \overline{z} \leq a$, which is indeed the case).

Question 4

Question: For the surface $\vec{r}(u,v)$, the vector element of surface is given by $d\vec{S} = \vec{r}_u \times \vec{r}_v \, du \, dv = \widehat{n} \, dS$, where $\widehat{n}$ is the unit normal to the tangent plane. With this convention, calculate, for the hemisphere $\mathcal{R}$ of radius $a > 0$ centered at the origin with $z \geq 0$, using spherical polar coordinates (take $u = \theta$, $v = \varphi$):

(a) $\iint_{\mathcal{R}} \vec{r} \bullet \widehat{n} \, dS$

(b) $\iint_{\mathcal{R}} |\vec{r}|^{-3} \vec{r} \bullet \widehat{n} \, dS$

Solution:

We'll assume the question wants outward flux (i.e. $d\vec{S}$ points outward). There are several ways to solve this problem, one of which is to directly compute the integrals. However, a simpler solution is to apply Gauss's law to the integral in part (b) (see the Tutorial 4 slides) and then find the integral in part (a) using this result. This works as follows: for the integral in part (b), we have

$$\begin{aligned} \iint_{\mathcal{R}} |\vec{r}|^{-3} \vec{r} \bullet \widehat{n} \, dS &= \iint_{\mathcal{R}} \left(\frac{\vec{r}}{|\vec{r}|^3} \right) \bullet \widehat{n} \, dS \\ &= 4\pi/2 \\ &= 2\pi \end{aligned}$$

This answer comes by observing that by Gauss's law, the flux of $\frac{\vec{r}}{|\vec{r}|^3}$ through the sphere of radius a (or, for that matter, through any other closed surface enclosing the origin) is 4π. We have to divide this by two because we only want the flux through one half of the sphere, which gives us 2π. We can then use this to find the integral in (a) by observing that $|\vec{r}| = a$ on the surface of the hemisphere (since the hemisphere has a radius of a):

$$\begin{aligned} \iint_{\mathcal{R}} \vec{r} \bullet \widehat{n} \, dS &= \iint_{\mathcal{R}} \vec{r} \left(\frac{|\vec{r}|^3}{|\vec{r}|^3} \right) \bullet \widehat{n} \, dS \\ &= \iint_{\mathcal{R}} \vec{r} \left(\frac{a^3}{|\vec{r}|^3} \right) \bullet \widehat{n} \, dS \\ &= a^3 \iint_{\mathcal{R}} \left(\frac{\vec{r}}{|\vec{r}|^3} \right) \bullet \widehat{n} \, dS \\ &= 2\pi a^3 \end{aligned}$$

Therefore, assuming an outward-directed normal vector $\widehat{n}$, the integral in part (a) evaluates to $2\pi a^3$, while the one in part (b) equals 2π.